# The Telescope Conjecture as Galois Theory

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November 7, 2023

# Classical Galois Theory

- Let $\text{Vect}_{\mathbb{Q}}$ be the category of finite-dimensional vector spaces over $\mathbb{Q}$.
- We have some trouble when studying $\text{Vect}_{\mathbb{Q}}$, because $\mathbb{Q}$ is not algebraically closed, so endomorphisms need not have eigenvalues.
- Consider a finite Galois extension $F/\mathbb{Q}$ with Galois group $G = \text{Aut}(F)$.
- For $V \in \text{Vect}_{\mathbb{Q}}$ we have $W = F \otimes V \in \text{Vect}_F$.
  This has a $\mathbb{Q}$-linear action of $G$ with $W^G = V$.
- For $W \in \text{Vect}_F$ and $g \in G$ define $g^*W$ to be the same abelian group but with $F$-action twisted by $g$.
- Given coherent identifications $g^*W \simeq W$ for all $g \in G$,
  we can construct $V \in \text{Vect}_{\mathbb{Q}}$ with $W \simeq F \otimes V$.
- In $\infty$-category framework:
  $G$ acts on $\text{Vect}_F$, and the map $\text{Vect}_{\mathbb{Q}} \to \text{Vect}_F^{hG}$ is an equivalence.
- This gives a map $K(\mathbb{Q}) \to K(F)^{hG}$, which is close to being an equivalence (Lichtenbaum-Quillen conjecture).
- We can define $\phi\colon F \otimes_{\mathbb{Q}} F \to \text{Map}(G, F)$ by $\phi(a \otimes b)(g) = a\, g(b)$.
- This is an isomorphism (the "tensor formula").

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- For a map $f \colon X \to Y$ of spectra with $\infty$-categorical kernel ($=$ homotopy fibre) $F$, we have a short exact sequence $\mathrm{cok}(\pi_{k+1}(f)) \to \pi_k(F) \to \ker(\pi_k(f))$.
- So $\pi_k(F)$ mixes the kernel and cokernel: not usually what we want.
- Consider instead the cosimplicial object

$$W \rightrightarrows \mathrm{Map}(G, W) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathrm{Map}(G^2, W) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathrm{Map}(G^2, W) \qquad \cdots$$

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- The (homotopy) inverse limit is $W^{hG}$, the homotopy fixed points. Here $\pi_k(W^{hG}) = H^k(G; W)$, which is 0 for $k > 0$ in Galois context.
- Using $F \otimes F = \mathrm{Map}(G, F)$ we see that the cosimplicial object is

$$W \rightrightarrows F \otimes W \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; F^{\otimes 2} \otimes W \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; F^{\otimes 3} \otimes W \qquad \cdots$$

  (and this is like an Adams resolution).
- For $G = \langle g \rangle \simeq \mathbb{Z}$, we just have $W^{hG} = \mathrm{fib}(g - 1)$ (as $BG \simeq S^1$).

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# Morava $K$-theory

- Fix $n > 0$. Over $\overline{\mathbb{F}_p}$, there is a formal group law $F_0$ with $[p]_{F_0}(x) = x +_{F_0} \cdots +_{F_0} x = x^{p^n}$. Any two such are isomorphic.
- There is an even periodic ring spectrum $K(n)$ such that $K(n)_0 = \overline{\mathbb{F}_p}$, and the associated formal group $\text{spf}(K(n)^0(\mathbb{C}P^\infty))$ corresponds to $F_0$ as above. This is *Morava $K$-theory*.
- This works up to homotopy but cannot be well rigidified: there are strictly associative versions but they are not canonical, and there are no strictly commutative versions.
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- Let $F$ be a finite field of order $p^n$ (so $F \simeq \mathbb{F}_p^n$ additively). Then there is a ring $WF$, isomorphic to $\mathbb{Z}_p^n$ additively, with $WF/p \simeq F$ as rings.
- This is the *Witt ring* of $F$; it is unique up to canonical isomorphism.
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- It can be shown that $K(n)_0 E(n) = C(\Gamma(n), \overline{\mathbb{F}_p})$. Similarly, $C(\Gamma(n), E(n)_0)$ is the completion of $E(n)_0 E(n)$ with respect to the ideal $I = (p = u_0, u_1, \dots, u_{n-1})$.
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- There is a short exact sequence $\mathrm{Aut}(F_0) \rightarrowtail \Gamma(n) = \mathrm{Aut}(E(n)) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}$. We call $\Gamma(n)$ the *Morava stabiliser group*.
- It can be shown that $K(n)_0 E(n) = C(\Gamma(n), \overline{\mathbb{F}_p})$. Similarly, $C(\Gamma(n), E(n)_0)$ is the completion of $E(n)_0 E(n)$ with respect to the ideal $I = (p = u_0, u_1, \dots, u_{n-1})$.
- This mean that $E(n)$ is a kind of Galois extension of $S$, but it is not faithful. Instead $E(n)^{h\Gamma(n)}$ is the $K(n)$-local sphere $L_{K(n)}S$.
- There is a sense in which $E(n)$ is the algebraic closure of $L_{K(n)}S$, so $\Gamma(n)$ is the absolute Galois group.

# Westerland $K$-theory

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- We put $R(n) = E(n)^{h\Gamma_1(n)}$ and call this *Westerland $K$-theory*. This is a faithful Galois extension of $L_{K(n)}S$ with Galois group $\Gamma(n)/\Gamma_1(n) = \mathbb{Z}_p^\times$.
- If we try to take $n = 0$ then there are various technical differences but morally $R(0) = \mathbb{Q}(\mu_{p^\infty})$.
- If $n = 1$ then $R(1)$ is the $p$-adic completion of complex $K$-theory and the action of $\mathbb{Z}_p^\times$ is by Adams operations.
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# Interaction with $K$-theory

- For a commutative group $U$ and a commutative ring $A$, there is an easy adjunction $\mathrm{CommRings}(\mathbb{Z}[U], A) \simeq \mathrm{CommGrp}(U, GL_1(A))$.
- For a commutative topological group $U$ and a commutative ring spectrum $A$, there is a similar adjunction between morphisms $\Sigma^\infty_+ U \to A$ of commutative ring spectra and morphisms $U \to GL_1(A)$ of $E_\infty$-spaces.
- For a commutative ring spectrum $A$, we have a $K$-theory spectrum $K(A)$.
- If $\mathcal{F}(A)$ is the category of finitely generated free $A$-modules and isos, then there is a canonical map $\Sigma^\infty_+ B\mathcal{F}(A) \to K(A)$.
- By restricting to the subcategory $\{A\} \subseteq \mathcal{F}(A)$ we get a ring map $\Sigma^\infty_+ BGL_1(A) \to K(A)$ or a map $BGL_1(A) \to GL_1(K(A))$ of spaces.
- By the construction of $TR(n)$ we have $\Sigma^\infty_+ B^n C_{p^\infty} \to TR(n)$ giving $K(TR(n)) \leftarrow \Sigma^\infty_+ B^{n+1} C_{p^\infty} \to TR(n+1)$.
- **Theorem:** We have a commutative diagram:

$$\begin{array}{ccc} \{T(n)\text{-local rings}\} & \xrightarrow{L_{T(n+1)}(K(-))} & \{T(n+1)\text{-local rings}\} \\ \downarrow{\scriptstyle L_{T(n)}(-\wedge TR(n))} & & \downarrow{\scriptstyle L_{T(n+1)}(-\wedge TR(n+1))} \\ \{T(n)\text{-local rings}\} & \xrightarrow[L_{T(n+1)}(K(-))]{} & \{T(n+1)\text{-local rings}\} \end{array}$$

- We discussed commutative ring spectra, but parts work more generally.

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- There is a spectrum $BP\langle n\rangle$ with $\pi_*(BP\langle n\rangle) = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ and $|v_k| = 2(p^k - 1)$.
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