SOME CONFORMAL TRANSFORMS

Write $t = u + jv$ and $z = x + jy$, where u, v, x and y are real. Let A be the region in the t-plane where $v > 0$, and let B be the region in the z-plane where $-1 < x < 1$ and $y > 0$.

I claim that there is an invertible conformal mapping from A to B given by $z = 2\arcsin(t)/\pi$, or $t = \sin(\pi z/2)$. This can either be seen directly, or using the general framework of Schwartz-Christoffel transforms.

For the direct argument, consider the map $f\colon B \to \mathbb{C}$ given by

$$t = f(z) = \sin(\pi z/2) = \sin(\pi x/2)\cosh(\pi y/2) + \cos(\pi x/2)\sinh(\pi y/2)j.$$

When $z \in B$ we have $|x| < 1$ (so $\cos(\pi x/2) > 0$), and also $y > 0$ (so $\sinh(y) > 0$). We conclude that $v = \mathrm{Im}(t) = \mathrm{Im}(f(z)) > 0$, so the point $t = f(z)$ lies in A, so we actually have a function $f\colon B \to A$. The bottom edge of B is the real interval $[-1, 1]$, or equivalently, the interval where $y = 0$ and $-1 \le x \le 1$. By inspecting the graph of f, we see that f carries this interval to itself in an invertible manner. Next, we have

$$\begin{aligned} f(-1 + jy) &= -\cosh(\pi y/2) \\ f(+1 + jy) &= +\cosh(\pi y/2). \end{aligned}$$

The left hand edge of B is the line where $x = -1$ and $y > 0$. By inspecting the graph of cosh, we see that $\cosh(\pi y/2)$ runs from 1 to ∞ along this line, so f carries the left edge of B to the interval $(-\infty, -1]$ in an invertible manner. Similarly, it carries the right hand edge to $[1, \infty)$. Putting all this together, we see that f carries the boundary of B invertibly to the real axis, which is the boundary of A. The general theory of conformal mappings now allows us to conclude that f carries B invertibly to A.

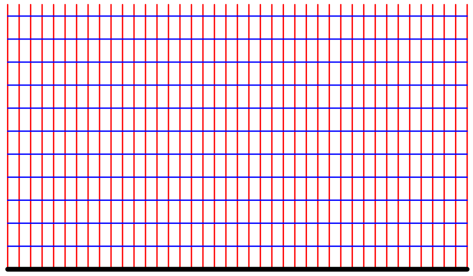
Alternatively, Schwartz-Christoffel theory says that there is an invertible conformal mapping between A and B, such that

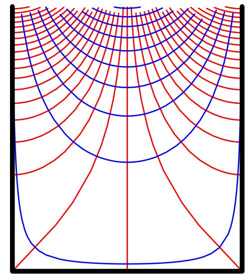
$$\frac{dz}{dt} = K(t - t_0)^{-1/2}(t - t_1)^{-1/2},$$

for some constants K, t_0 and t_1 with t_0 and t_1 real. We are free to make an additional transformation of the form $t \mapsto \alpha t + \beta$ (with α and β real and $\alpha \neq 0$). Using this, we may assume that $t_0 = -1$ and $t_1 = +1$, and thus that $(t-t_0)(t-t_1) = (t + 1)(t - 1) = t^2 - 1$. This gives

$$\begin{aligned} z &= K \int \frac{dt}{\sqrt{t^2 - 1}} \\ &= Kj \int \frac{dt}{\sqrt{1 - t^2}} \\ &= Kj \arcsin(t) + c. \end{aligned}$$

When $t = 1$ this gives $z = Kj\pi/2$, but the point $t = t_1 = 1$ must correspond to the corner of B at $z = 1$, so we have $1 = Kj\pi/2 + c$. By considering $t = -1$ in the same way, we see that $-1 = -Kj\pi/2 + c$. Ading these two equations gives $c = 0$, and it then follows that $Kj = 2/\pi$, so $z = 2\arcsin(t)/\pi$ as claimed.



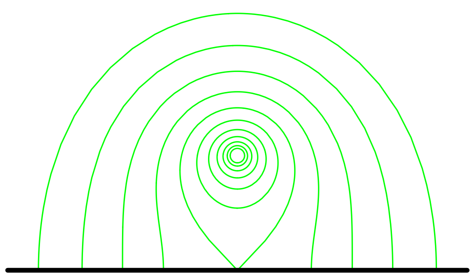


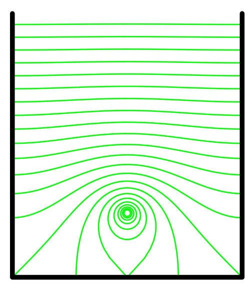
The picture on the left shows the t-plane, with lines of constant u marked in red, and lines of constant v in blue. The picture on the right shows the corresponding curves in the z-plane.

Now consider a current flowing in a wire passing perpendicularly through our diagram at the point $t = j$, and suppose that the boundaries of the regions A and B are made of infinitely permeable iron. The method of images says that the resulting magnetic field in A is the same as the field coming from two equal currents at j and $-j$, with no iron boundary. The complex potential for this field is $\omega = \log((t-j)(t+j)) = \log(t^2+1)$ (or a positive multiple of this). The magnetic flux function is thus

$$\phi = \mathrm{Re}(\omega) = \mathrm{Re}(\log(t^2+1)) = \log|t^2+1|.$$

The field lines are the lines $\phi = c$ for constant c. The equation $\phi = c$ is equivalent to $|t^2+1| = e^c$, or $t^2+1 = e^{c+i\theta}$ for some θ, so $t = (1 - e^{c+i\theta})^{1/2}$. This makes it easy to plot the field lines in the t-plane, as shown on the left below.





As the two-dimensional magnetic field equations are conformally invariant, the field lines in the z-plane are obtained by just applying the rule $z = 2\arcsin(t)/\pi$ to the field lines in the t-plane. The result is shown on the right in the diagram. Note that the current is located at the point

$$z = 2\arcsin(j)/\pi = (2\,\mathrm{arcsinh}(1)/\pi)j \simeq 0.56j.$$