Maths with Maple — Week 2 lab session

Solving equations

Maple has powerful facilities for solving equations, but they are not perfect. Often one has to rearrange an equation in some way before Maple will succeed in solving it. It is quite common for Maple to give a complicated solution, when a slightly different approach might give a simpler one. Thus, solving equations is more of an art than a science.

Exercises 1.1, 1.4, 2.1 and 4.2 appear on the online test for this week.

Numbers in square brackets refer to the "Maple reference" notes, which were distributed in the first lecture, and are available online at `http://www.shef.ac.uk/nps/MAS100`.

1. Algebraic equations

Exercise 1.1. Enter

```
f := (x) -> 2*x^4-2222*x^3+224220*x^2-2222000*x+2000000;
```

to define the function

$$f(x) = 2x^4 - 2222x^3 + 224220x^2 - 2222000x + 2000000.$$

(See [**10.1**– **10.6**] for more about the syntax used here for defining functions.) Then solve $f(x) = 0$ (as explained in [**8.1**]) to find the roots. Use the answer to factorize $f(x)$, and then enter `factor(f(x));` to check your answer.

Exercise 1.2. Define $y = x^4 - x^3 - x^2 - x/8 + 1/64$ and $z = 16x^3 - 24x^2 - 6x + 2$. (Here we do not use the arrow notation, because we do not have a (x) on the left hand side.) Ask Maple to find the values of x where $y = 0$; you should get four solutions. It is best to use a command like `sols := solve(...)` so that you can reuse the solutions later; now `sols[1]` will refer to the first solution, `sols[2]` will refer to the second, and so on. Now look more closely at the solutions. You should see that they all involve the same four terms, but with different plus and minus signs. Examine the pattern of signs carefully. Can you rephrase Maple's answer more neatly? Next, read [**6.1**– **6.2**]. Use the `subs()` command to find the value of z at the four points where y is zero. You can do this as four separate steps, or (better) you can read [**14.1**] and use the `seq()` command to do it in one go. (There are some interesting phenomena going on in this calculation, that will eventually be explained in the level 4 course on Galois Theory.)

Exercise 1.3. As general background, you should be aware that the last two exercises were chosen carefully to work out nicely. For a typical polynomial there may be no formula for the roots, and even if there is, it may be very ugly. Moreover, even if there is a formula and the roots are all real numbers, the formula may still involve complex numbers as an intermediate stage. For an example of this, set `_EnvExplicit:=true;` again, and then ask for the roots of $x^3 - 3x + 1$. Notice that the answer involves I (the square root of minus one) in several places.

In a case like this, it is more useful to find numerical approximations to the roots instead of a massive exact formula. Read [**8.7**], and ask Maple to find an approximate solution to the equation $y = 0$, where $y = x^3 - 3x + 1$. It is also helpful to plot the graph (say for $-2 \le x \le 2$) and see where the roots lie: read [**11.1**] for help with this. From either of these approaches, we see that there is precisely one negative root for this equation. Use [**12.1**] and [**6.1**] to find the value of dy/dx at this negative root.

Exercise 1.4. Enter the definition

$$g(x) = \frac{b^2 - c^2 + (1 + c^2)x}{1 - c^2 + c^2 x}.$$

We will study the fixed points of g, or in other words the values where $g(x) = x$.

(a) Ask Maple to solve the equation $g(x) = x$ for x. (Remember to use the syntax in [**10.1**] when entering the definition of g.) You should get two solutions.
(b) For some special value(s) of b and/or c, Maple's answer does not make sense. Go back to the definition of g and the equation $g(x) = x$, and work out by hand what happens in those special cases.
(c) For which value(s) of b and/or c are Maple's two solutions actually the same?
(d) Give a self-contained summary of your conclusions, that could be read and understood by someone who had not attempted the question.

2. Approximate solutions

Exercise 2.1. Define $f(x) = \sin(\pi x + e^{-x})$ (remembering that e^{-x} is `exp(-x)` and π is `Pi`). If you ask Maple to solve this, it will give an answer in terms of an obscure function called `LambertW`. [1] We will ignore this for the moment, and look instead for a numerical approximation to the roots, concentrating on the case where $x \ge 0$.

[1] If you are curious, you can enter `?LambertW` or visit `http://mathworld.wolfram.com/LambertW-Function.html`.

(a) First plot the graph of $f(x)$, say from $x = 0$ to $x = 10$. Roughly where are the roots? Can you explain why they are where they are?

(b) Enter `fsolve(f(x)=0,x);`. This finds a root at about $x = -0.55$, ignoring all the roots with $x \geq 0$ that we saw in the graph. To find a root near $x = 2$ instead, enter

```
fsolve(f(x)=0,x=2);
```

To find roots near $x = 1$, $x = 2$, $x = 3$ and so on, up to $x = 10$, enter this:

```
seq(fsolve(f(x)=0,x=n),n=1..10);
```

(c) Now define $r(n) = n - e^{-n}/\pi - e^{-2n}/\pi^2$ (using syntax as in [**10.3**]). In (a) and (b) we saw that for every integer n, there is a root that is close to $x = n$. I claim that this root is even closer to $x = r(n)$. To see this, enter the definition of r and then

```
seq(evalf(r(n)),n=1..10);
```

Compare this with your final answer in (b).

3. Infinite families of solutions

Exercise 3.1. Consider the equation $\tan(\pi x)^2 = 3$.

(a) To find the solutions graphically, plot the function $\tan(\pi x)^2 - 3$ for a reasonable range of values of x. (Because $\tan(\pi x)$ blows up to infinity for certain values of x, it is necessary to cut down the vertical range to get a meaningful picture [**11.2**], and it is also useful to ask Maple to calculate extra points to make the graph more accurate [**11.7**]. You should see that there are many roots in the picture, repeating in a regular way, so there are actually infinitely many roots if we allow x to run from $-\infty$ to $+\infty$.

(b) Click with your mouse on each of the places where the graph crosses the x-axis; approximate coordinates will then be shown in a small box at the top left of the Maple window. You should be able to guess the exact coordinates from this.

(c) Ask Maple to solve the equation $\tan(\pi x)^2 = 3$ for x. Maple will report only two solutions, although we have seen that there are really infinitely many.

(d) You should have seen in (b) that the solutions are all of the form $x = n + 1/3$ or $x = m - 1/3$, where n and m are integers. To persuade Maple to find this answer, we have to enter `_EnvAllSolutions:=true;`, and then ask Maple again to solve the equation. It will respond as follows:

$$\{x = 1/3 + \_Z1\}, \{x = -1/3 + \_Z2\}$$

This is the same as our answer, except that Maple uses the symbols $\_Z1$ and $\_Z2$ (instead of n and m) for arbitrary integers.

4. Linear equations

Exercise 4.1. Consider the equations $x/2 + y/3 + z/4 = 1$, $x/3 + y/4 + z/5 = 2$, $x/4 + y/5 + z/6 = 3$. To solve these, we enter

```
eqns := {x/2 + y/3 + z/4 = 1,x/3 + y/4 + z/5 = 2,x/4 + y/5 + z/6 = 3};
sols := solve(eqns,{x,y,z});
```

It would also work to do all this in one step:

```
solve({x/2+y/3+z/4=1,x/3+y/4+z/5=2,x/4+y/5+z/6=3},{x,y,z});
```

However, this can make things cramped and hard to organise, especially if our equations are just one piece of a more complex problem. Now find the value of $x^2 + y^2 + z^2$ at the point where the above equations are satisfied.

Exercise 4.2. Solve the following systems of equations by hand. You should find that

- one system has no solutions
- one system has a single, fully-determined solution
- one system has a solution in which one of the variables is free to take any value; this means that there are infinitely many different solutions.

(a) $p + q + r = 0$ $\quad$ $p + 2q + 3r = 1$
(b) $u + v = 1001$, $\quad$ $u + 2v = 1002$, $\quad$ $u + 3v = 1006$
(c) $x + y + z = 2$, $\quad$ $x + 2y + 3z = 2$, $\quad$ $x + 4y + 9z = 2$

When you have found the solutions by hand, find them again using Maple [**8.1,8.3**]. Note that Maple returns an equation like $w = w$ to indicate that w may take any value. Note also that when there are no solutions, Maple prints nothing at all, which can be disconcerting. If this worries you, you can enclose the `solve` command in square brackets. For example, `solve({x=1,2*x=1},{x});` gives a completely empty response, but `[solve({x=1,2*x=1},{x})];` gives a response of [], so you can at least see that Maple has actually done something.