

# Differentiation 1

Various parts of questions 1, 2 and 3 will be on the online test.

**Exercise 1.** On the course home page ([www.shef.ac.uk/nps/MAS100](http://www.shef.ac.uk/nps/MAS100)), click on the link marked `defineq.mws`. This should open up a worksheet containing a single line starting `q:=sscanf("f*6...`. Click on this line and press RETURN. This will set up a “mystery function” called  $q(x)$ , whose properties we will investigate.<sup>1</sup>

- Plot  $q(x)$ . Experiment to find a range of  $x$  values that shows all the main features of the graph.
- For which values of  $x$  do we have  $q'(x) < 0$ ? (You can answer this by just looking at the graph.) How about  $q'(x) = 0$  or  $q'(x) > 0$ ?
- Enter `Q:=(x,h)->(q(x+h)-q(x))/h;`, so that  $q'(x)$  is the limit of  $Q(x, h)$  as  $h$  tends to zero [10.5]. Put `e:=exp(1);` for convenience [7.3]. Tell Maple to do all numerical calculations to 30 digits [3.3].
- Enter `Q(e^2,0.1)` to get an approximate value for  $q'(e^2)$ . Replace 0.1 by a smaller number to get a better approximation. What do you think is the exact value of  $q'(e^2)$ ?
- Work out  $q'(e^{-2})$  and  $q'(e^4)$  in the same way. Guess the formula for  $q'(e^t)$ .
- Suppose that  $x > 0$ . What is the number  $t$  such that  $x = e^t$ ? Deduce a formula for  $q'(x)$  (valid for  $x > 0$ ).

**Exercise 2.** We will call the following *Rolle's principle*: between any two roots of a function  $f(x)$ , there is at least one root of  $f'(x)$ .<sup>2</sup>

- How can you find the roots of  $f'(x)$  by looking at the graph of  $f(x)$ ?
- Define  $f(x) = x^3 - x$  using [10.1]. Plot the graph. You should see that Rolle's principle is correct in this case. We will now analyse this more precisely using Maple (although it would be easy to do it by hand).
  - Ask Maple [8.2] to find the roots of  $f(x)$ .
  - Ask Maple [12.1,12.4] to find  $f'(x)$ . (Note that `f'(x)` is **not** valid Maple syntax.)
  - Ask Maple to find the roots of  $f'(x)$ .
  - Plot  $f(x)$  and  $f'(x)$  together, and observe how the roots of  $f(x)$  alternate with the roots of  $f'(x)$ .
- Repeat part (b) for the function  $g(x) = \sin(x) + \sin(3x)/3$ . Maple will give you six roots for  $g(x)$ , but four of them are complex numbers, so they can be ignored. It will just give three real roots for  $g'(x)$ . However, both  $g(x)$  and  $g'(x)$  are periodic, so they really have infinitely many roots. You should work out from the graph where the rest of the roots lie. (You could ask Maple to do this by setting `_EnvAllSolutions:=true;`, but it does not do a good job: the answer comes out in a complicated and confusing form.)
- Can you explain roughly why Rolle's principle is true? (Part (a) is relevant.)
- Consider  $h(x) = \tan(x)$ . Is Rolle's principle true in this case? How does this relate to your answer to (d)?

**Exercise 3.** Suppose that  $y$  is a function of  $x$ , and we write  $y' = dy/dx$  and so on. The *schwartzian derivative* of  $y$  is defined to be

$$S(y) = y'''/y' - \frac{3}{2}(y''/y')^2.$$

- Work out  $S(x^n)$  by hand. For which  $n$  is  $S(x^n) = 0$ ? For which  $n$  is  $S(x^n)$  undefined?
- Enter the definition into Maple:

```
S := (u) -> diff(u,x,x,x)/diff(u,x) - (3/2)*(diff(u,x,x)/diff(u,x))^2;
```

Now use Maple to check your answer to (a).

- Define  $y = (ax + b)/(cx + d)$  (where  $a, b, c$  and  $d$  are constants), then ask Maple to calculate and simplify  $y', y'', y'''$  and  $S(y)$ . You should find that  $S(y) = 0$ . In fact, functions of this form are the *only* ones for which  $S(y) = 0$ .
- Now define  $z = (ap^x + b)/(cp^x + d)$  (where  $p$  is yet another constant) and simplify  $S(z)$ . You should find that  $S(z)$  is a constant, not depending on  $x$ . However, some creativity is required to persuade Maple to do the right simplifications. Here is one approach that works: expand out  $1/S(z)$ , simplify it, and then take 1 over the result.
- Now define

$$T(u) = (u')^{1/2}((u')^{-1/2})''.$$

<sup>1</sup>For a non-mathematical challenge, you can enter `?option` and `?interface`, read the resulting help pages carefully, and work out how to print out a more comprehensible definition of  $q(x)$ . You will then need to read more help pages to understand the definition.

<sup>2</sup>As we will see, there are some exceptions. However, there is a more precisely formulated version, called *Rolle's Theorem*, for which there are no exceptions (that is what “theorem” means).

Enter this definition in Maple (with syntax similar to part (b)). Choose some functions  $w$  at random (eg  $w = \ln(x)$ , or  $w = \sin(x)^2$ ) and calculate  $S(w)$  and  $T(w)$ . What do you notice? Can you prove it?

**Exercise 4.** Consider the function  $y = \cos(-2\ln(x))$  (defined for  $x > 0$ ).

- (a) Plot the graph from  $x = 0$  to  $x = 0.01$ , and then from  $x = 0$  to  $x = 1000$ .
- (b) One of the roots of  $y$  is at  $x = a_0 = e^{-\pi/4}$ . Let  $m_0$  be the gradient  $dy/dx$  at  $x = a_0$ . Find and simplify  $m_0$ .
- (c) What is the equation of the line  $L_0$  that passes through  $(a_0, 0)$  with gradient  $m_0$ ? Plot this line together with the graph of  $y$ , for  $x$  from 0 to  $2a_0$ .
- (c) What are the other roots of  $y$ ? Repeat part (b) for several other roots. What do you notice about the lines  $L_i$ ? Can you prove it?