Maths with Maple — Week 6 lab session

Differentiation 2

Parts of exercises 1, 2, 3, 5 and 6 will appear on the online test.

1. Maxima and minima

Exercise 1. Plot the function $y = x^n e^{-x}$ for various values of n. Solve $dy/dx = 0$ and so find the maximum value of y (assuming $x > 0$).

Exercise 2. Consider the function

$$p(x) = -10x^6 + 156x^5 - 945x^4 + 2780x^3 - 4080x^2 + 2880x.$$

Plot the graph for a suitable range. Find all the critical points of $p(x)$ (where $p'(x) = 0$). Which of these are inflection points (where $p''(x) = 0$ as well)? What is the maximum value of $p(x)$, and for what value of x does it occur?

2. Implicit derivatives

Exercise 3. Put $u = x\sin(x^2 + y^2) - y\cos(x^2 + y^2)$. Plot the curve where $u = 0$, using the `implicitplot` command **[11.11]** . Remember that you need `with(plots):` to make this work **[11.18]**. A reasonable range is to let x and y run from -5 to 5, and you should ask Maple to plot extra points **[11.12]** to get a decent picture. The curve is called a *Fermat spiral*.

Now find the slope of the curve using the `implicitdiff` command **[12.5]**, and call the answer `slope1`. Maple will give the answer in a form involving $\sin(x^2 + y^2)$ and $\cos(x^2 + y^2)$. Can you use the relation $u = 0$ to rewrite `slope1` in a form that does not involve sin or cos?

We next claim that the curve can also be described parametrically by $(x, y) = (t\cos(t^2), t\sin(t^2))$. In checking this, we will use the symbol `xt` for "x in terms of t", and thus enter

```
xt := t*cos(t^2);  yt := t*sin(t^2);
```

Use the `subs` command **[6.1]** to substitute `xt` for `x` and `yt` for `y` in u, and simplify the result. You should get zero, indicating that the point $(t\cos(t^2), t\sin(t^2))$ really does lie on the curve $u = 0$. (It is not very hard to do this by hand as well.) You can also enter `plot([xt,yt,t=-5..5]);` and see that you get the same picture as before.

Now calculate dx/dt and dy/dt, and so get a formula for dy/dx in terms of t. Call this one `slope2`. Use the `subs` command to rewrite `slope1` in terms of t (rather than x and y), simplify the result, and check that it is the same as `slope2`.

Exercise 4. Consider the curve with equation $u = 0$, where

$$u = (x^2 + y^2)^2 + 85(x^2 + y^2) - 500 + 18x(3y^2 - x^2).$$

It turns out that this can also be given parametrically by

$$(x, y) = (6\cos(t) + 8\cos(t)^2 - 4, 2\sin(t)(3 - 4\cos(t))).$$

Analyse this situation just as in the previous question: plot the graph in two different ways, find the slope in two different ways, and check that they are really the same.

3. Higher derivatives

Exercise 5. Enter the following definition into Maple **[10.3,12.3]**:

$$r(n) = \frac{d^n}{dx^n}\left(\frac{x^n \ln(x)}{n!}\right).$$

Find $r(n)$ for a reasonable range of numbers n, using the `seq(...)` command **[14.1]**. Then experiment to find a formula for $r(n) - r(n-1)$. Finally, deduce a formula for $r(n)$, of the form

$$r(n) = \text{something} + \sum_{k=1}^{n} (\text{something}).$$

Exercise 6. Put $y = t^2 e^t$. Simplify the expression

$$z = y''' + ay'' + by' + cy,$$

where a, b and c are constants. Hence find a, b and c such that y satisfies the differential equation

$$y''' + ay'' + by' + cy = 0$$

for all t.

Exercise 7. Put $p(n) = e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$. Enter this definition in Maple [**10.3,12.3**], and then calculate $p(n)$ for n from 1 to 10. (It is best to define $p(n)$ to be `sort(expand(exp(x^2) * ...))`, to get the answer in a convenient form. To print $p(1), \dots, p(10)$ on separate lines, enter `seq(print(p(n)),n=1..10)` rather than just `seq(p(n),n=1..10)`.)

Write down as many things as you can about the general form of $p(n)$, distinguishing between the case where n is even and the case where n is odd. Using these, predict as much as you can about $p(12)$, and then check your predictions.

In doing this question, you should see the numbers 2, 4, 8, 16, 32, 64 and so on, which you should recognise as powers of 2. You will also see the numbers 2, 12, 120, 1680, 30240, which are much less likely to be familiar. To see what they are, enter the sequence in the search box at `http://www.research.att.com/~njas/sequences/`. (The response is long and complicated, but the formula that you need appears in the first few lines.)

It turns out that $p(n)$ is closely related to the Hermite polynomials, which Maple calls `HermiteH(n,x)`. Enter `q:=(n)->simplify(HermiteH(n,x));`, then work out $q(n)$ for various values of n, then write down the precise relationship between $p(n)$ and $q(n)$.