Maths with Maple — Week 7 lab session

Integration of rational functions

Setup

Visit `http://www.shef.ac.uk/nps/MAS100` and click with your **right** mouse button on the link marked `ratint.mws`. Then select `Save Target As..`, save the worksheet somewhere, start up Maple separately, and then open the worksheet using the File menu. When the worksheet has started up, click the button on the Maple toolbar marked with a triple exclamation mark (!!!). This will execute all the commands in the worksheet, which does two things:

(a) It defines a command called `ratint()`. This works in essentially the same way as the usual integration command `int()`, but it gives the answer in a form which is slightly better for the questions on this sheet.

(b) It defines a number of rational functions $g_1(x), g_2(x), \ldots$ (which you should enter as `g[1](x)`, `g[2](x)` and so on). These can be used as examples later (when the question says "for various rational functions ...").

If you use `restart` in doing this problem sheet, you will lose the definition of `ratint()` and the functions $g_i(x)$. It is better to use `unassign` instead. Remember that the syntax is like `unassign('a','b')`, with single quote marks.

Questions

Your task is to answer the following questions, using a mixture of mathematical thinking and experimental calculations with Maple. When calculating integrals, use `ratint(...,x);` instead of `int(...,x);`.

Exercise 1. When x is large, the graph of $y = \int \left(\frac{x^3+1}{x^2+1}\right)^3 dx$ looks like $y = cx^n$ for some constant c and integer n. Find c and n.

Exercise 2. Let a, b, c and d be nonzero constants. The graph of $y = \int \frac{ax^2+b}{cx^2+d}\,dx$ looks like a straight line for large x. What is the slope of the line?

Exercise 3. Are the following true or false?

(a) Integrals of rational functions sometimes involve terms like $a\ln(x^2+ux+v)$, where a, u and v are constants.

(b) Integrals of rational functions sometimes involve terms like $x\ln(x+u)$, where u is constant.

(c) Integrals of rational functions sometimes involve terms like $a\ln(x+u)^2$, where a and u are constants.

Exercise 4. For various different values of b and c (which may be positive, negative or zero), do the following:

(i) Plot $y = x^2 + bx + c$.

(ii) Calculate $b^2 - 4c$.

(iii) Find $\int \frac{dx}{x^2+bx+c}$, and observe whether it involves the functions arctan or ln.

What is the relationship between (i), (ii) and (iii)?

Exercise 5. For various rational functions $g(x)$, do the following:

(i) Find $\int g(x)\,dx$, and look for any terms like $\ln(|x-u|)$ (but ignore terms like $\ln(x^2+px+q)$).

(ii) Plot $g(x)$ together with $\int g(x)\,dx$. (You will generally need to specify the vertical range in order to get a useful picture.)

How are the numbers in (i) related to the pictures in (ii)?

Exercise 6. For various rational functions $g(x)$, do the following:

(i) Enter `d:=denom(factor(g(x)));` to find the denominator of $g(x)$ (in other words, the term on the bottom when $g(x)$ is written as a single fraction).

(ii) Plot d.

(iii) Find $\int g(x)\,dx$, and look for terms like $a/(x-u)$ or $a/(x-u)^n$.

How are the numbers u and n in (iii) related to the answers to (i) and (ii)?

Now enter `r:=randpoly(x)/randpoly(x);` to generate a random rational function, and then `ratint(r,x);` to integrate it. Now go back in the worksheet to the line `r:=randpoly(x)/randpoly(x);` and press ENTER twice, to generate and integrate a new example. Repeat this many times. You should see that you only get ln() and arctan() terms, together with multiples of x^n for some small values of n. Can you see why terms like $a/(x-u)$ hardly ever occur?