Maths with Maple — Week 10 lab session

Taylor series

1. Taylor series

Exercise 1.1. We will study the Taylor series at $x = 0$ of the function

$$y = \ln\left(\sqrt{\frac{1+x}{1-x}}\right).$$

This has the form $y = \sum_{k=0}^{\infty} a_k x^k$, where a_k is $1/k!$ times the value of $d^k y/dx^k$ at $x = 0$. You should start by entering the definition of y.

(a) Find **[12.3]** and simplify d^5y/dx^5. Then put $x = 0$ using the `subs` command **[6.1]**, and divide by 5! to get a_5.
(b) Find a_1, a_2, a_3 and a_4 in the same way.
(c) To do this more efficiently, enter

```
a := (n) -> subs(x=0,diff(y,x$n))/n!;
```

then use the `seq` command **[14.1]** to calculate $a(1), \ldots, a(5)$ in one go. (This syntax does not work properly when $n = 0$, but it is easy to see that $a_0 = 0$ anyway.)
(d) Now enter `add(a(k)*x^k,k=1..12);` to get the 13'th order Taylor series for y at $x = 0$. Then do the same thing more easily using the `series` command **[12.7]**.
(e) Guess the complete Taylor series for y at $x = 0$.

Exercise 1.2. (a) Find the 12th order Taylor series for $\sin(x)$ at $x = 0$, using the `series` command **[12.7]**. Convert the answer to an ordinary polynomial as in **[12.8]**. Call the result s.
(b) Plot s and $\sin(x)$ together **[11.5]** for x from -8 to 8, with the vertical range also restricted **[11.2]** from -10 to 10. You should see that the two graphs are very close together for x between -5 and 5, but that they diverge very rapidly outside that range.
(c) Now define $t(n)$ to be the n'th order Taylor series for the function $\sin(x)$ at $x = 0$. (Use syntax like `t:=(n)->convert(series(...)...)`). Plot $t(n)$ and $\sin(x)$ together (with the same ranges as in (b)) for various n. As you make n bigger, the two graphs will get closer together. How big must n be for the two graphs to look identical? What then happens if you expand the horizontal range to run from -10 to 10?
(d) To plot $\sin(x)$ together with $t(2), t(6), t(10), t(14), \ldots, t(42)$, enter

```
plot([sin(x),seq(t(4*n+2),n=0..10)],x=0..20,-2..10);
```

Now modify this to get a good picture of $t(4), t(8), \ldots, t(44)$. Why do we not bother with $t(n)$ for odd n?
(e) Now define $r(n)$ to be the n'th order Taylor series for $\cos(x)$ at $x = 0$. Expand out $t(10)^2 + r(10)^2$, and call the result q. What should this be equal to, approximately? How could you check this?

Exercise 1.3. Find the Taylor series for $y = \frac{x(1+x)}{(1-x)^3}$ at $x = 0$, to some reasonably high order. You should be able to guess from this that

$$y = \sum_{k=1}^{\infty} (\text{something})x^k.$$

You can ask Maple to confirm this by entering

```
sum((something) * x^k,k=1..infinity);
```

You should see that Maple's answer is the same as y (rearranged slightly).

Exercise 1.4. Enter `series(sin(x),x=Pi/2,12)` to find the 12th order Taylor series of $\sin(x)$ around $x = \pi/2$. How is this related to the 12th order Taylor series of $\cos(x)$ around $x = 0$? Why?

2. Maple syntax revision

The questions in this section all cover things you have done before, but this time there are no hints about the syntax. Look at the notes on Maple (`http://www.shef.ac.uk/nps/MAS100/notes/primer.pdf`) and/or the earlier lab sheets if there are things that you do not remember.

Exercise 2.1. Find $\cos(\ln(\pi + 20))$ to 20 decimal places.

Exercise 2.2. Enter the definition $f(x) = x^2 - 29/16$. Ask Maple to work out $f(0)$. If it just gives you "$f(0)$" back, then you used the wrong syntax to define f; read Section 10 of the Maple notes. What is $f(f(f(-1/4)))$?

Exercise 2.3. Using the various commands for manipulating algebraic expressions, find out the relationship between

$$a = (1+x)^5 - 3(1+x)^4 + 5(1+x)^3 - 3(1+x)^2 + 3(1+x) + 3.$$

and

$$b = (7x^2 - 6x - x^8)/(x-1)^2.$$

Exercise 2.4. Find the coefficient of x^6 in the expression

$$\left(\frac{(x^{12}-1)(x^2-1)}{(x^6-1)(x^4-1)}\right)^{10}.$$

(You will need to simplify the expression first.)

Exercise 2.5. The equations

$$\begin{aligned} x^2 + y^2 + z^2 &= 9 \\ (x-1)^2 + (y-1)^2 + (z-1)^2 &= 2 \\ 4x^2 + yz &= 2x(y+z) \end{aligned}$$

have only one solution in which x, y and z are all integers. What is that solution?

Exercise 2.6. Find all the (infinitely many) solutions to the equation $\sin(\theta)^2 = 3\cos(\theta)^2$.

Exercise 2.7. Find the solution to the equations $x^2 - y^2 = 2xy = 1$ for which x and y are real numbers and $x > 0$. Your answer should not involve the word `RootOf`.

Exercise 2.8. Find an approximate solution to $x = \log(x+20)$ close to $x = 3$.