# Maths with Maple — Week 11 lab session

You can start with Section 1 or Section 2, whichever you prefer.

## 1. Miscellaneous problems

**Exercise 1.1.** Plot the curve given by $x = \cos(t)/2 - \cos(2t)/4$ and $y = \sin(t)/2 - \sin(2t)/4$, together with the circle of radius $1/4$ centred at $(-1, 0)$, and two circles of radius 0.0945 centred at the points $(-0.1225, \pm 0.7449)$. Display all these in the same picture with the same scale in both directions, and with no axes. (This is a basic outline of the Mandelbrot set. Google will find you much prettier pictures.)

**Exercise 1.2.** Plot all the curves $|x|^n + |y|^n = n^{n/2}$ in the same picture for $n = 1, \dots, 9$. Use the `seq()` command to avoid excessive typing, make sure that you use a range that includes all nine curves, and add an option to make Maple draw a more accurate picture.

**Exercise 1.3.** Use the `listplot` command to plot the values $50^n/n!$ for $n$ from 1 to 100. Mark these values as separate points, not connected by lines.

**Exercise 1.4.** There is a certain function $\zeta(z)$ defined for complex numbers $z$, called the Riemann zeta function. It features in the most famous open problem in all of pure mathematics, called the Riemann Hypothesis: is it true that whenever $s$ and $t$ are positive real numbers with $\zeta(s + it) = 0$, we have $s = 1/2$? There is a great deal of evidence for this but no proof. If true, it will have important consequences for the distribution of prime numbers (and perhaps therefore for cryptography).

(a) Enter `with(MTM)` to load the definition of $\zeta(z)$.
(b) Enter `f:=(t)->Re(zeta(1/2+I*t))`, to define $f(t)$ to be the real part of $\zeta(1/2 + it)$. Define $g(t)$ to be the imaginary part in the same way.
(c) Plot the graphs of $f(t)$ and $g(t)$ together. Note that $\zeta(1/2 + it)$ is zero when both graphs cross the horizontal axis in the same place. What is the smallest positive value of $t$ where this happens?
(d) Now instead plot the curve given parametrically by $x = f(t)$ and $y = g(t)$ (so $(x, y)$ corresponds to $\zeta(1/2 + it)$ in the Argand diagram). In this picture, zeros of the $\zeta$-function correspond to the places where the curve passes through the origin. You should see that there are many of them (if you use a suitable range for $t$).
(e) Now change the $1/2$ to some other value and repeat steps (c) and (d). The Riemann hypothesis predicts that you should not find any zeros. Thus, the two graphs in (c) will never crosss the horizontal axis at the same time, and the curve in (d) will never pass through the origin. Check this.

**Exercise 1.5.** Put $y = \exp(-1/x)$, for $x > 0$. Investigate the behaviour of $y$ and the derivatives $d^k y/dx^k$, by calculating formulae and plotting graphs.

## 2. Solitons

In this exercise we will investigate the behaviour of some interesting functions called solitons, which arise in the theory of water waves in a shallow channel. Put

$$q = \sqrt{2} \qquad p = \log(3 + 2q)$$
$$r = x - 4t \qquad s = q(x - 8t)$$
$$T = 32\cosh(2r - p) + 16\cosh(2s - p) + 16 \qquad B = 4(1 + q)\cosh(r)\cosh(s) + (4q - 8)\exp(r + s)$$
$$\phi_0 = 2\cosh(r)^{-2} \qquad \phi_1 = 4\cosh(s)^{-2}$$
$$\phi_2 = 2\cosh(r - p)^{-2} \qquad \phi_3 = 4\cosh(s - p)^{-2}$$
$$\phi_4 = T/B^2.$$

Note that the functions $\phi_i$ depend $x$ (position) and also $t$ (time). We can therefore differentiate them with respect to $x$ or with respect to $t$. In this situation where there are several variables, it is traditional to use the notation $\partial\phi_i/\partial x$ and $\partial\phi_i/\partial t$ rather than $d\phi_i/dx$ and $d\phi_i/dt$. These can be entered in Maple in the usual way, as `diff(phi[i],x)` and `diff(phi[i],t)`.

Check that the functions $\phi_i$ all satisfy the Korteweg-de Vries equation:

$$\frac{\partial\phi_i}{\partial t} + \frac{\partial^3\phi_i}{\partial x^3} + 6\phi_i\frac{\partial\phi_i}{\partial x} = 0.$$

Maple will do this quite happily for $i = 0, \ldots, 3$. For $\phi_4$, however, you need to help by telling Maple to convert all hyperbolic functions to explicit exponentials at the beginning. The easiest thing is to enter `phi[4] := convert(phi[4],exp);` before working out the derivatives.

Next, do some plots to investigate the behaviour of these functions. One approach is to use the `subs()` command to put $t = 1$ (say), and then plot the result for $-30 \leq x \leq 30$. You can then change $t$ and do the same again. It is illuminating to plot several of the functions $\phi_i$ in the same graph.

You can also make a movie. For example:

```
with(plots):
animate(
 plot,
 [[phi[2],phi[3]+5,phi[4]+10],x=-30..30],
 t=-3..3,
 frames=100, scaling=constrained, axes=none
);
```

This makes Maple work quite hard, so it may take a minute or so. When the plot appears, click on it; then some playback controls will appear near the top of the Maple window, which should be fairly self-explanatory. Note that we have plotted $\phi_2$, $\phi_3 + 5$ and $\phi_4 + 10$; the $+5$ and $+10$ terms just shift the graphs of $\phi_3$ and $\phi_4$ upwards, so we can see them more clearly.

Experiment with further movies of this type, including functions like $\phi_0 + \phi_2$ as well as the functions $\phi_i$ themselves. See if you can write a clear, self-contained summary of how the functions $\phi_i$ are related to each other.

Next, the *momentum* of $\phi_i$ is defined to be the integral $M_i = \int_{-\infty}^{\infty} \phi_i\, dx$. As $\phi_i$ depends on $t$ as well as $x$, you might think that $M_i$ depends on $t$. However, it turns out that the dependence on $t$ cancels out, so $M_i$ is a constant. Ask Maple to calculate $M_i$ for $i = 0, \ldots, 3$, and you will see that this is true.

Maple is not clever enough to calculate $M_4$ symbolically. However, it can do it numerically if we tell it to fix a particular $t$ (say $t = 0$) and give it a hint about the best numerical method to use:

```
M[4] := int(subs(t=0,phi[4]),x=-infinity..infinity,numeric,method=_Gquad);
```

(If we leave out `method=_Gquad` then Maple will use a different method which gives the same answer but takes several minutes, whereas the method above takes only a few seconds.) You should then check that $M_4 = M_0 + M_3$.

We also define the *energy* of $\phi_i$ to be the integral $E_i = \int_{-\infty}^{\infty} \phi_i^2\, dx$. Repeat the above steps for the energy, giving exact answers for $E_0, \ldots, E_3$ and an approximate answer for $E_4$, and check that $E_4 = E_0 + E_3$.