# Special functions

December 10, 2009

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- You should know how these are defined in terms of the primary functions (for example, $\sinh(x) = (\exp(x) - \exp(-x))/2$, and $\sec(x) = 1/\cos(x)$)
- You should either remember the properties of the secondary functions, or be able to derive them from the properties of the primary functions

# The exponential function

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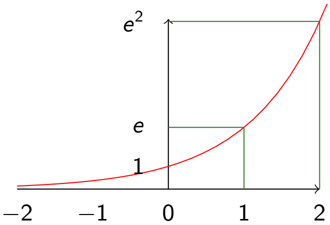

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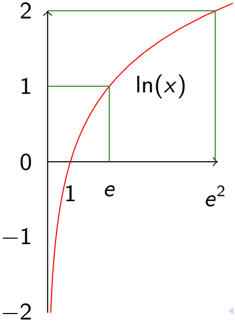
- 

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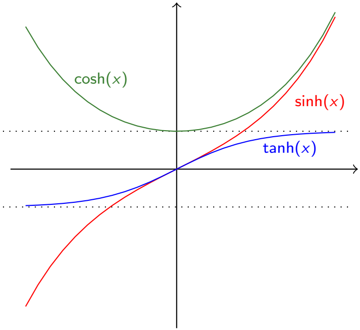
- 

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- Now put $v = e^y$, so $uv = e^{x+y}$.

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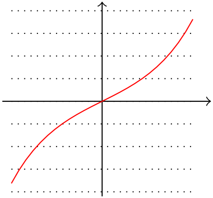

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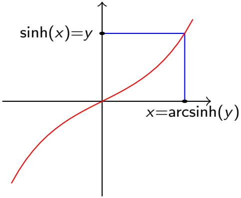

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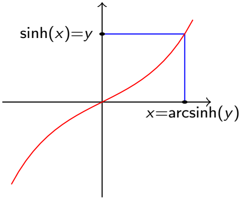


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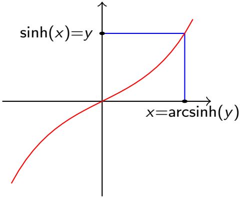


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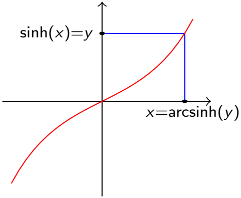


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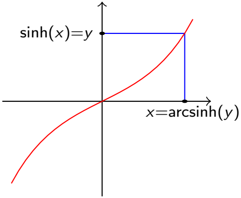


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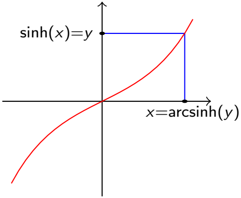


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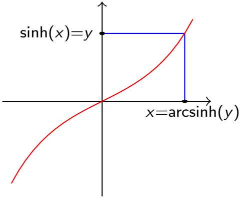


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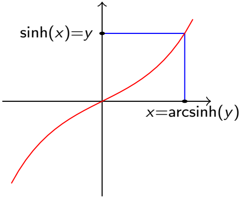


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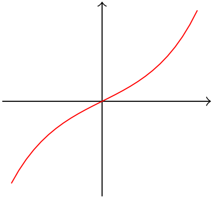
- Similarly, $\operatorname{arccosh}(y) = \ln(y + \sqrt{y^2 - 1})$, defined for $y \geq 1$
- and $\operatorname{arctanh}(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$, defined when $-1 < y < 1$.

# Graphs



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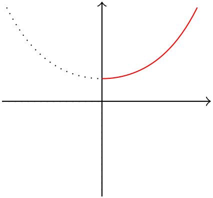

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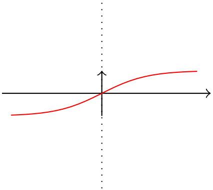

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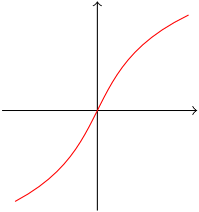

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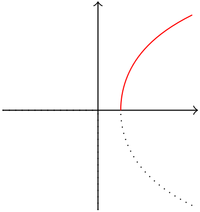

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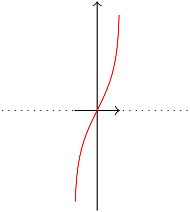

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