Special functions

December 10, 2009

Trigonometric functions

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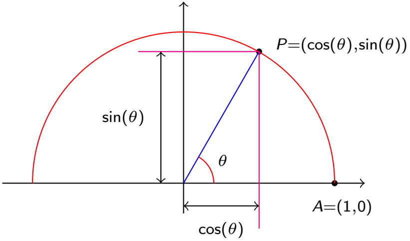

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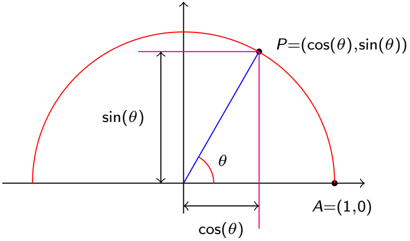

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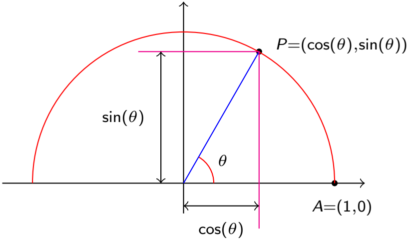

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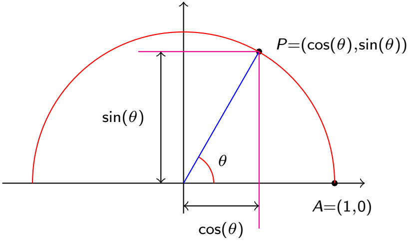

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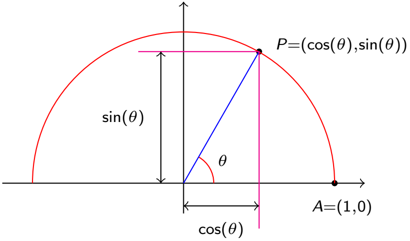

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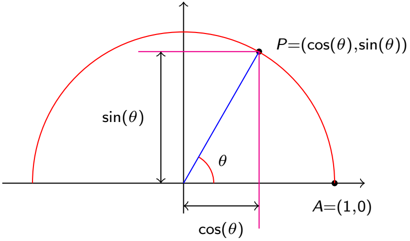

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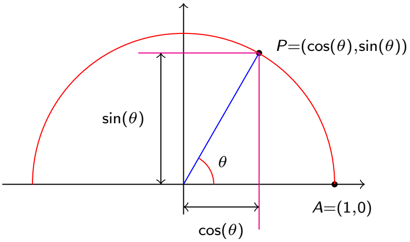

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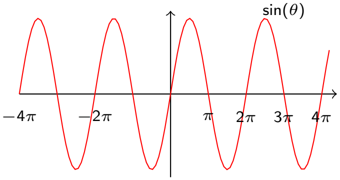
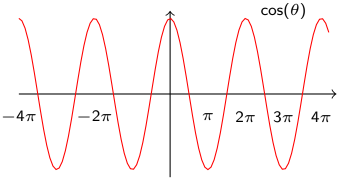
Graphs


$\sin(\theta)$
-4π
-2π
π
2π
3π
4π


$\cos(\theta)$
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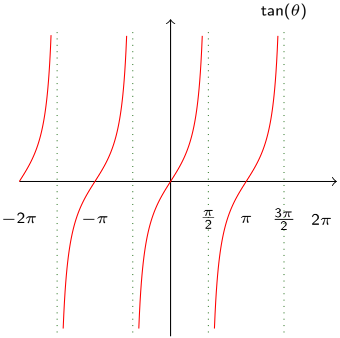
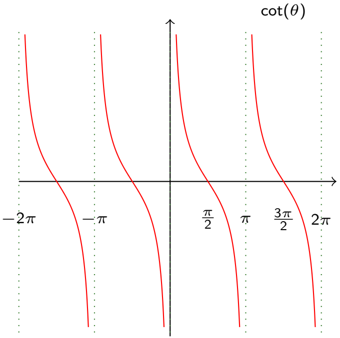

$\tan(\theta)$
-2π
$-\pi$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π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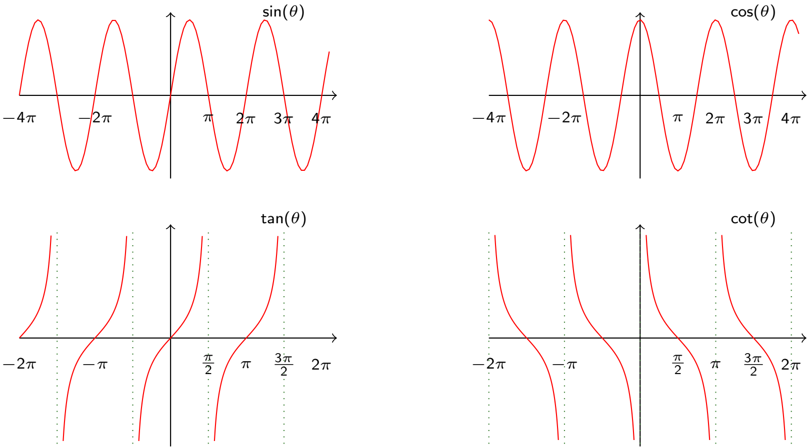

$$\begin{aligned} \sin(\pi/2+x) &= \cos(x) & \cos(\pi/2+x) &= -\sin(x) \\ \sin(\pi+x) &= -\sin(x) & \cos(\pi+x) &= -\cos(x) \\ \sin(2\pi+x) &= \sin(x) & \cos(2\pi+x) &= \cos(x) \\ \sin(-x) &= -\sin(x) & \cos(-x) &= \cos(x). \end{aligned}$$

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&= \cos(x) + \sin(x) i.
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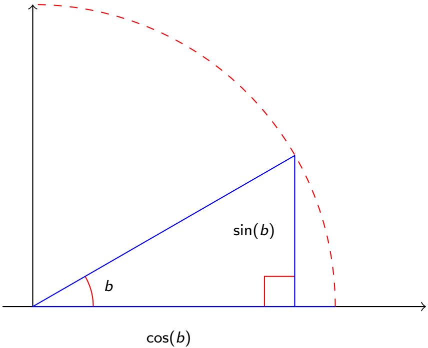

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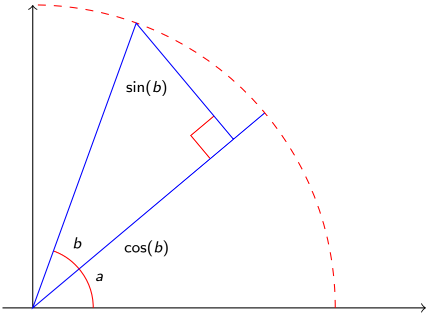

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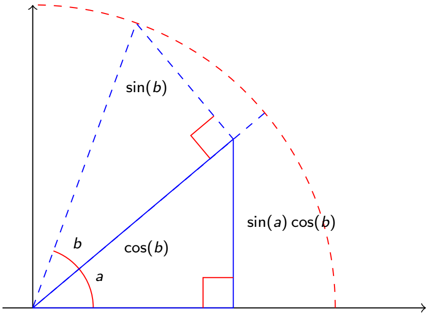

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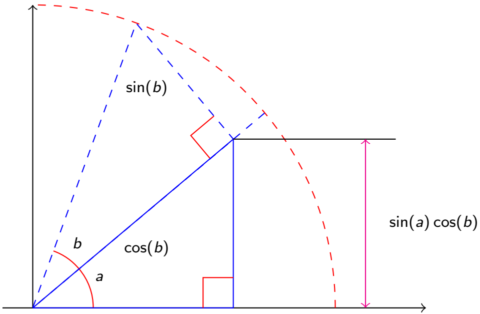

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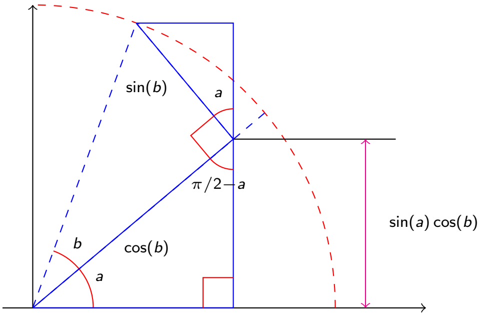

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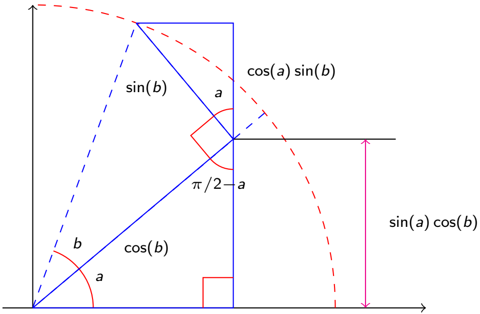

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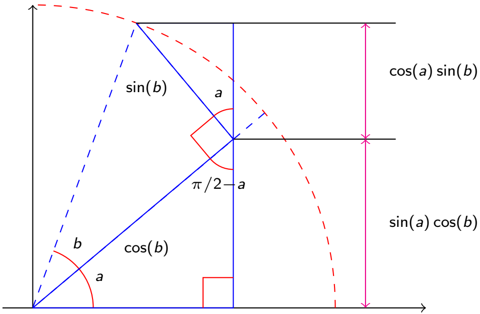

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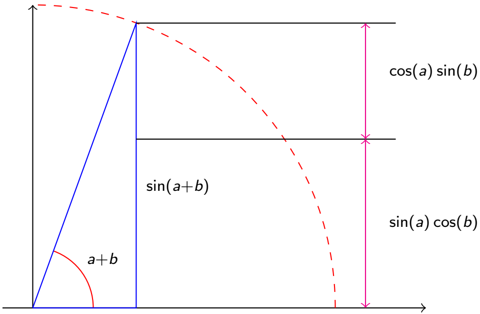

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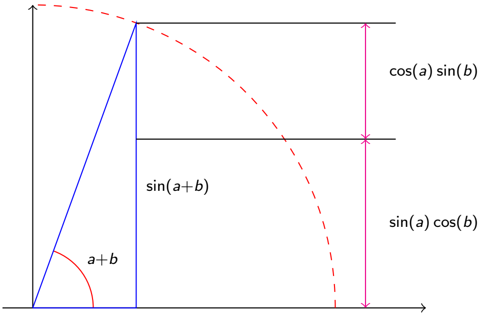

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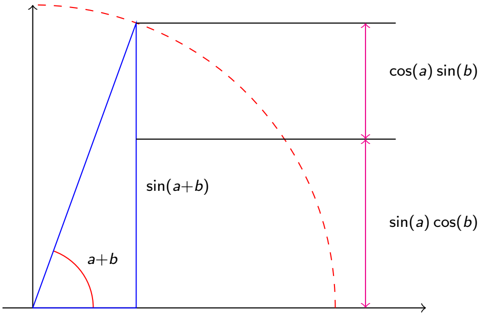

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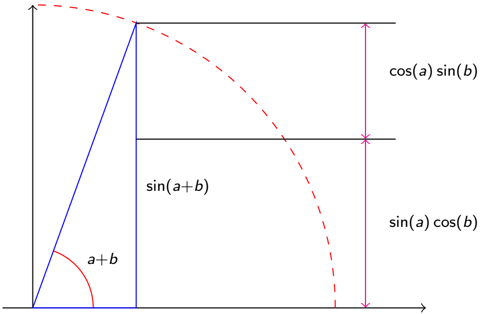

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Finite Fourier series

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- A *finite Fourier series* is a sum of constant multiples of functions of the form $\sin(nx)$ or $\cos(mx)$ (with $n, m \in \mathbb{Z}$). Note that the constant function $f(x) = a = a\cos(0x)$ is included.

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\sin(nx)\sin(mx) &= \tfrac{1}{2}\cos((n-m)x) - \tfrac{1}{2}\cos((n+m)x).
\end{aligned}$$

- **Method:** Rewrite using $\cos(n\theta) = (e^{in\theta} + e^{-in\theta})/2$ and $\sin(n\theta) = (e^{in\theta} - e^{-in\theta})/2i$, expand out, then rewrite using $e^{im\theta} = \cos(m\theta) + \sin(m\theta)i$.

Finite Fourier series

- A *finite Fourier series* is a sum of constant multiples of functions of the form $\sin(nx)$ or $\cos(mx)$ (with $n, m \in \mathbb{Z}$). Note that the constant function $f(x) = a = a\cos(0x)$ is included.
- The phrase *trigonometric polynomial* means the same thing.
- Many functions can be rewritten as finite Fourier series:

$$\begin{aligned} \sin(x)^2 &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \\ \sin(x)^3 &= \tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x) \\ \sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\ \sin(x)^4 + \cos(x)^4 &= \tfrac{3}{4} + \tfrac{1}{4}\cos(4x) \\ \sin(nx)\sin(mx) &= \tfrac{1}{2}\cos((n-m)x) - \tfrac{1}{2}\cos((n+m)x). \end{aligned}$$

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- Once a function has been rewritten in this form, it is very easy to differentiate it or integrate it.

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\end{aligned}$$

Special values

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You should know the following values of $\sin(\theta)$ and $\cos(\theta)$:

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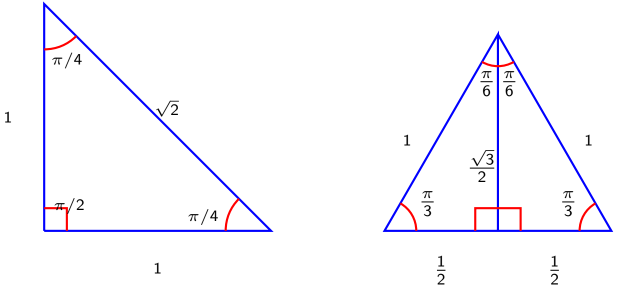
Proved by considering these triangles:



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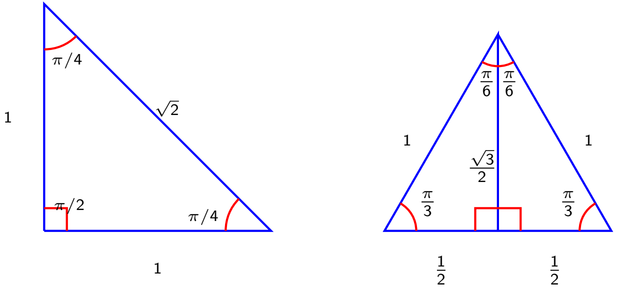
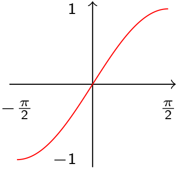
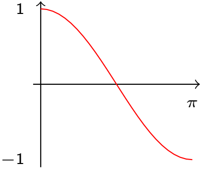


You should also be able to deduce things like $\cos(5\pi/6) = -\sqrt{3}/2$.

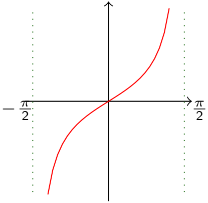
Inverse trigonometric functions



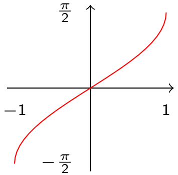
$\sin\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1,1]$



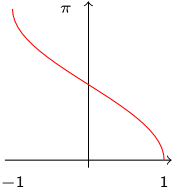
$\cos\colon [0,\pi] \longrightarrow [-1,1]$



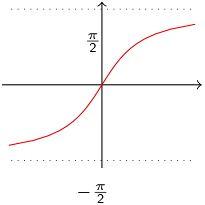
$\tan\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R}$



$\arcsin\colon [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\arccos\colon [-1,1] \longrightarrow [0,\pi]$



$\arctan\colon \mathbb{R} \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$