Differentiation

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You must learn to find derivatives quickly and accurately.

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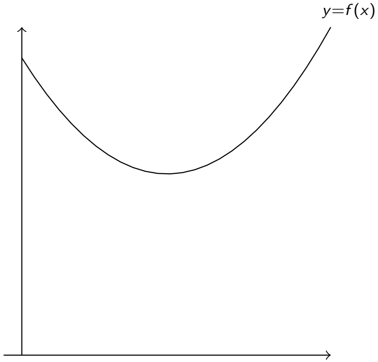
- We sometimes write y' for dy/dx (care needed).

Slopes



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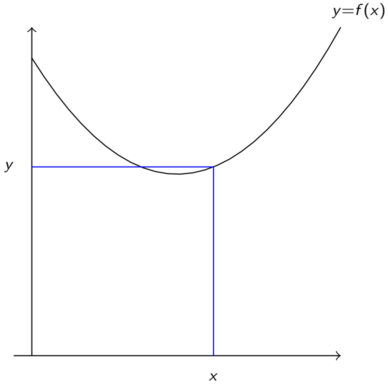
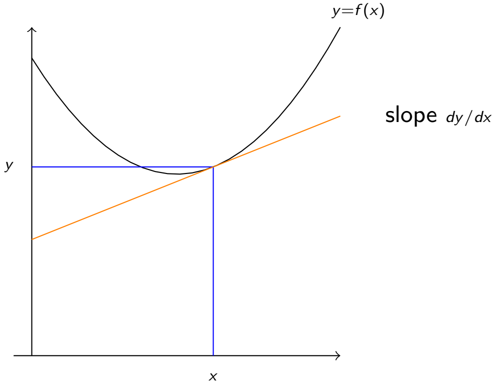


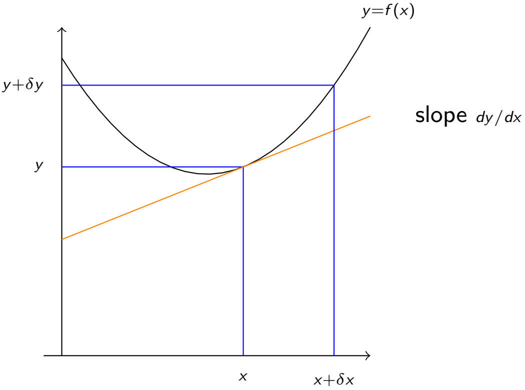
Consider variables x and y related by $y = f(x)$.

Slopes



dy/dx is the slope of the tangent line to the graph.

Slopes



If x changes by a small amount δx, then y will change by a small amount δy.

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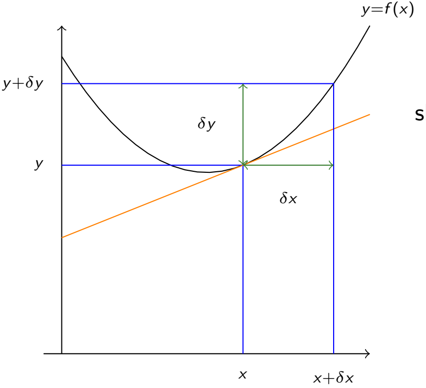

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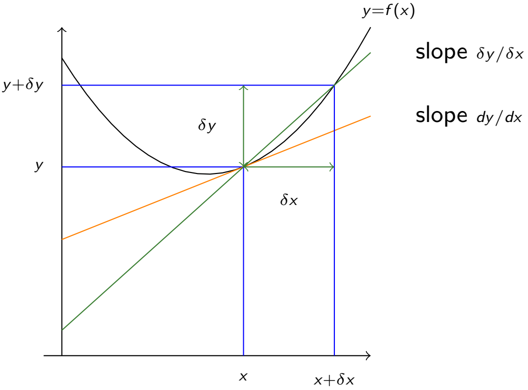
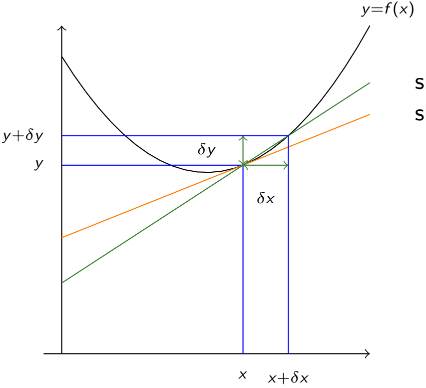


The ratio $\delta y / \delta x$ is the slope of a chord cutting across the graph.

Slopes

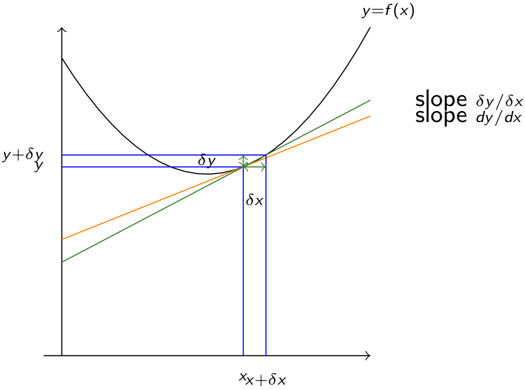


slope $\delta y / \delta x$

slope dy / dx

The slope of the chord changes slightly as δx decreases.

Slopes



As δx approaches zero, the chord approaches the tangent, and $\delta y/\delta x$ approaches dy/dx.

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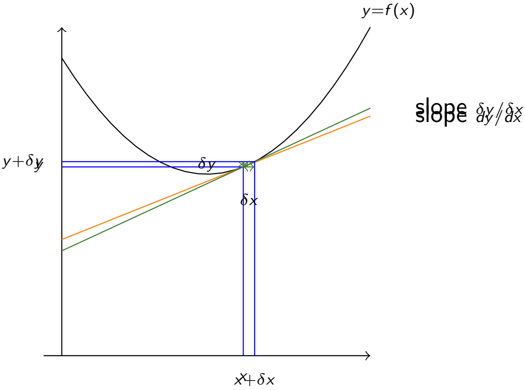

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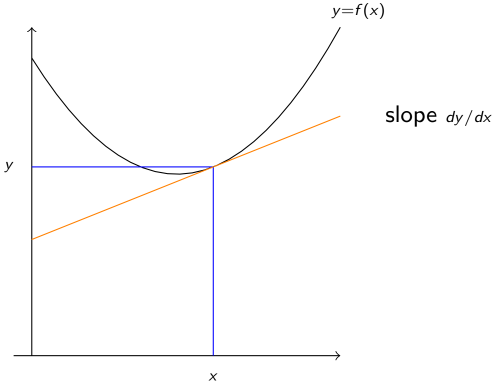

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- Similarly:

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1} \text{ for all } n.}$$

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  $$\begin{aligned} f'(x) &= \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \longrightarrow 0} e^x\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right) \\ &= e^x(1 + 0 + 0 + \cdots) \\ &= e^x. \end{aligned}$$

The exponential function

- Consider the function $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.
- $$f(x+h) - f(x) = e^{x+h} - e^x = e^x(e^h - 1) = e^x\left(h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right)$$

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- Conclusion: $\exp'(x) = \exp(x)$.

Special functions

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\exp'(x) &= \exp(x) \\
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\tanh'(x) &= \operatorname{sech}(x)^2 = 1 - \tanh(x)^2
\end{aligned}
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\log'(x) &= 1/x \\
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\operatorname{arccosh}'(x) &= (x^2-1)^{-1/2} \\
\operatorname{arctanh}'(x) &= (1-x^2)^{-1} \\
\arcsin'(x) &= (1-x^2)^{-1/2} \\
\arccos'(x) &= -(1-x^2)^{-1/2} \\
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- The inverse function rule also gives the remaining derivatives.

The product rule

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(The approximations become exact in the limit as $\delta x \to 0$.)

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- Consider variables u and v depending on x, and put $w = u/v$. Then

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$$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos(x)^2}$$

Examples of the quotient rule

$$\frac{d}{dx}\left(\frac{x}{\log(x)}\right) = \frac{1.\log(x) - x x^{-1}}{\log(x)^2} = \frac{\log(x) - 1}{\log(x)^2} = \log(x)^{-1} - \log(x)^{-2}$$

(Aside: $x/\log(x) \simeq$ (number of primes $\leq x$))

$$\frac{d}{dx}\left(\frac{x}{1 - x^2}\right) = \frac{1.(1 - x^2) - x.(-2x)}{(1 - x^2)^2} = \frac{1 - x^2 + 2x^2}{(1 - x^2)^2} = \frac{1 + x^2}{(1 - x^2)^2}$$

Now consider $\tan'(x)$, remembering that $\tan(x) = \sin(x)/\cos(x)$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) &= \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos(x)^2} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos(x)^2} \\ &= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2}\end{aligned}$$

Examples of the quotient rule

$$\frac{d}{dx}\left(\frac{x}{\log(x)}\right) = \frac{1.\log(x) - xx^{-1}}{\log(x)^2} = \frac{\log(x)-1}{\log(x)^2} = \log(x)^{-1} - \log(x)^{-2}$$

(Aside: $x/\log(x) \simeq$ (number of primes $\leq x$))

$$\frac{d}{dx}\left(\frac{x}{1-x^2}\right) = \frac{1.(1-x^2) - x.(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

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Examples of the quotient rule

$$\frac{d}{dx}\left(\frac{x}{\log(x)}\right) = \frac{1.\log(x) - xx^{-1}}{\log(x)^2} = \frac{\log(x) - 1}{\log(x)^2} = \log(x)^{-1} - \log(x)^{-2}$$

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