

Integration

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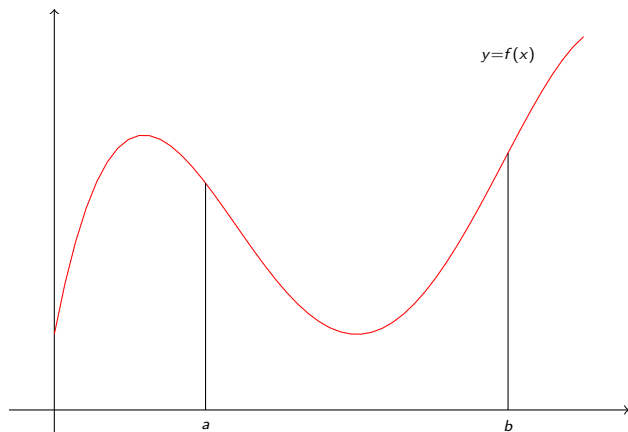
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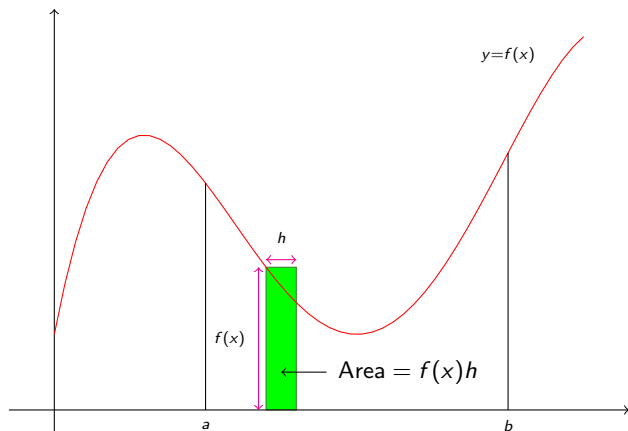
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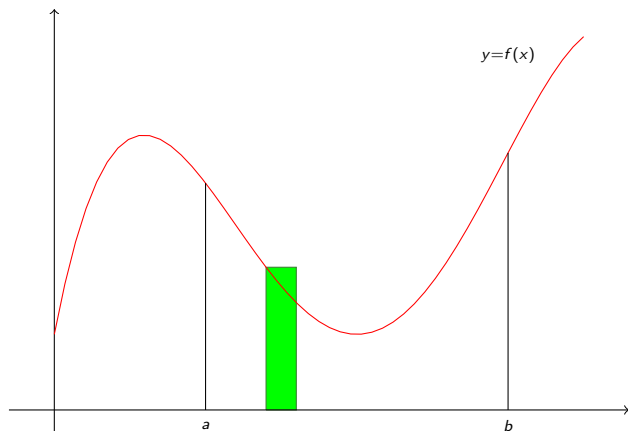
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- ▶ A current flowing in a wire exerts a magnetic force on a moving electron. There is a formula for the force contributed by a short section of wire; to get the total force, we integrate.



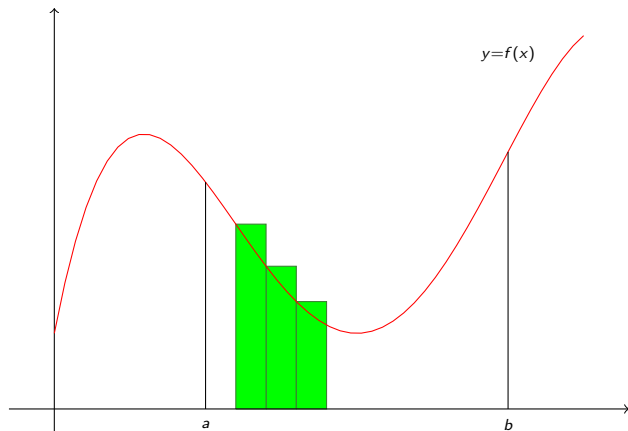
Consider the integral $\int_a^b f(x) dx$.



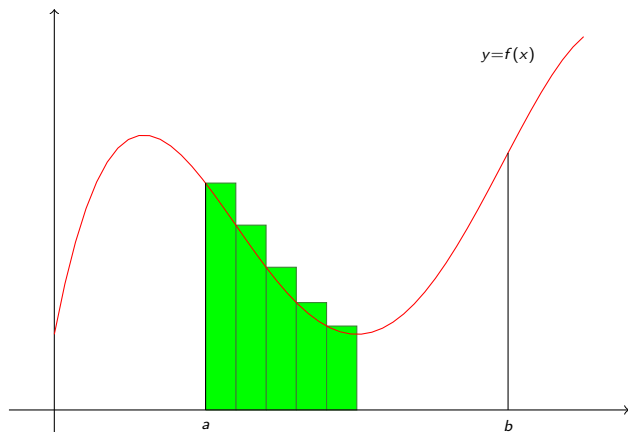
For each short interval $[x, x + h] \subset [a, b]$, we have a contribution $f(x)h$. This is the area of the green rectangle.



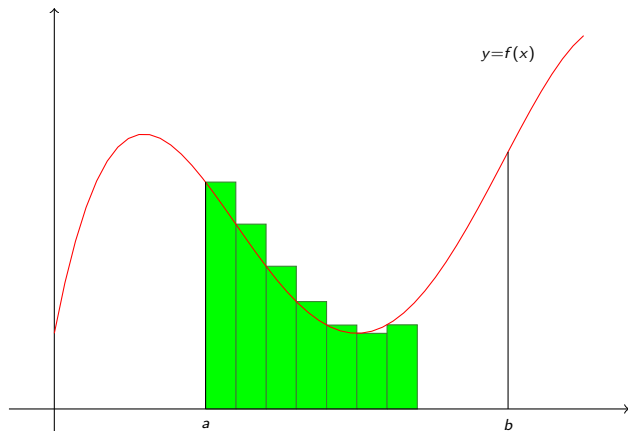
This is the contribution from one short interval, but we need to add together the contributions from many short intervals.



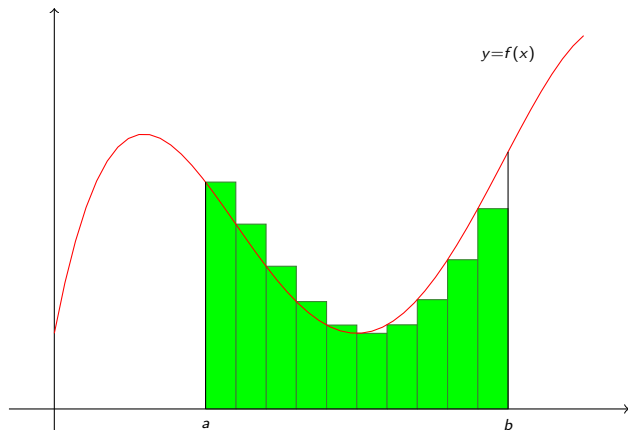
Here we have added in two more intervals



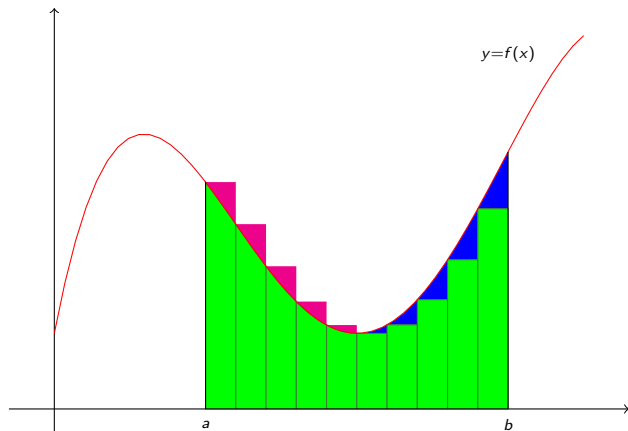
Here we have added in two more intervals – and two more



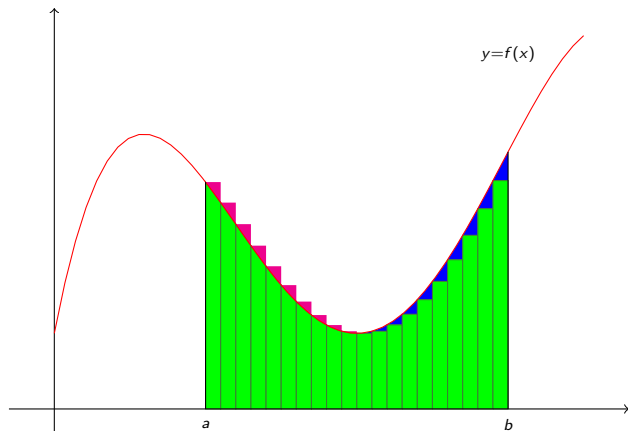
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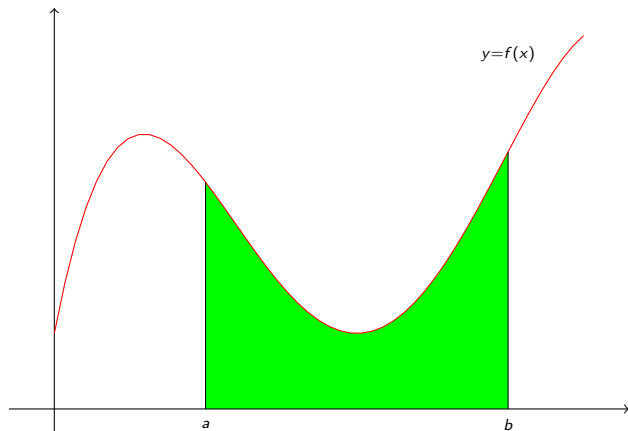
Now we have divided the whole interval $[a, b]$ into subintervals of length h . The sum of the terms $f(x)h$ is the area of the green region.



This is not exactly the same as the area under the curve, because of the regions marked in blue and pink.



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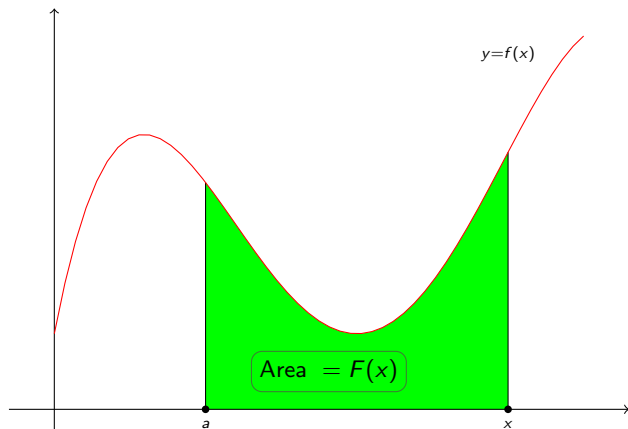
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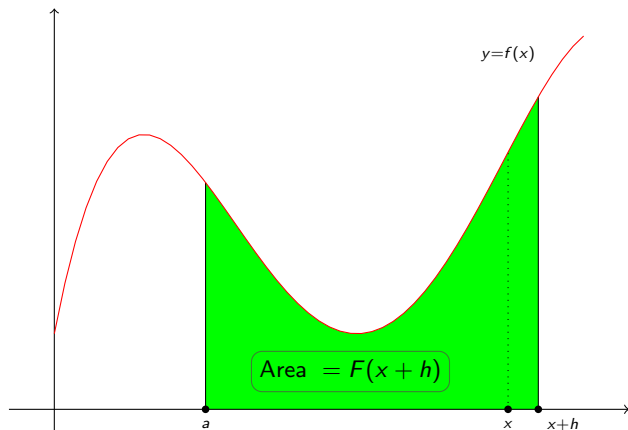
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- ▶ $\int_a^b \frac{1}{x} = [\log(x)]_a^b = \log(b) - \log(a)$

Proof of the Fundamental Theorem



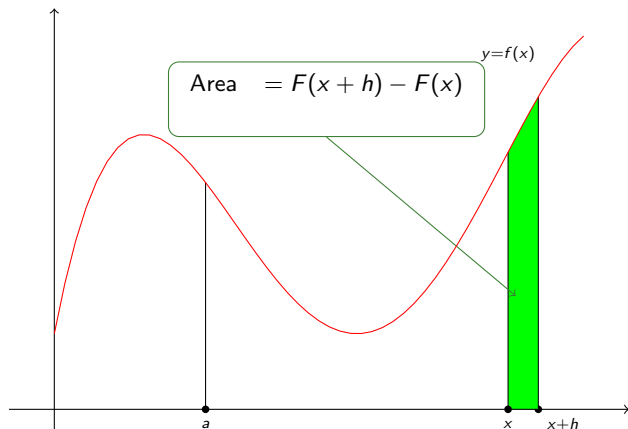
Choose a number a , and define $F(x) = \int_a^x f(t) dt$. We must show that $F'(x) = f(x)$.

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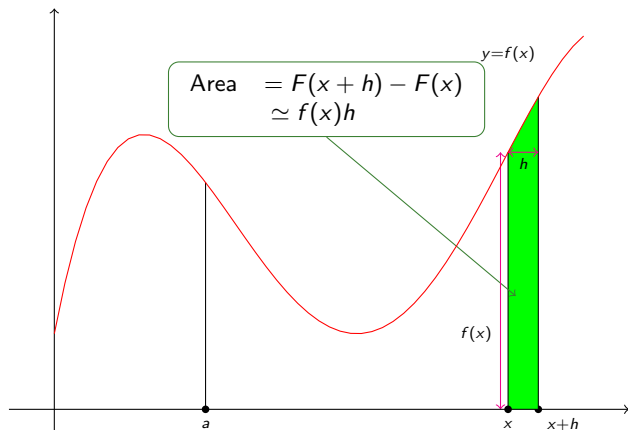
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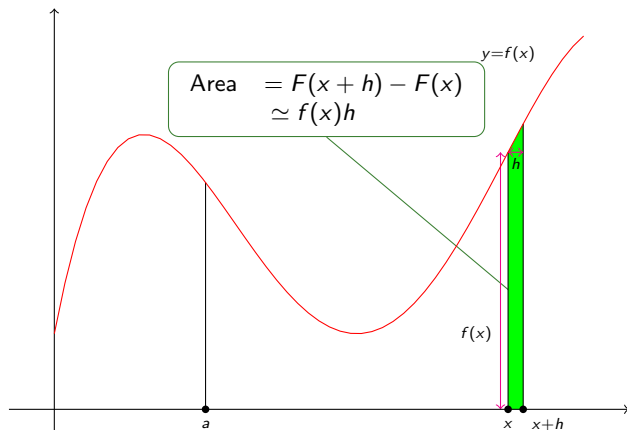
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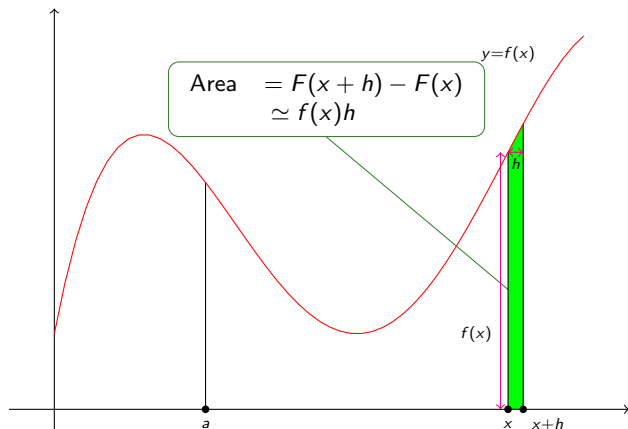
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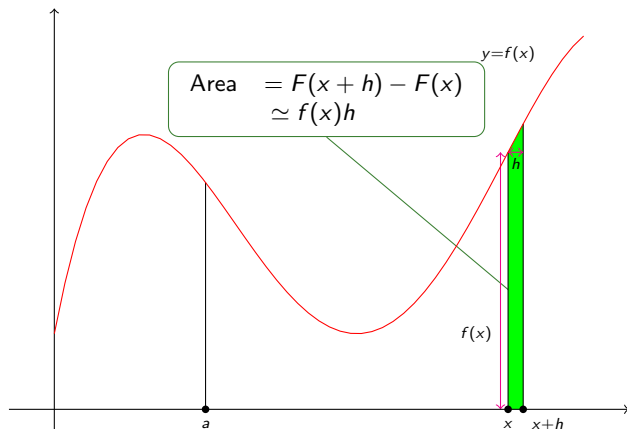
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- ▶ Maple's `int()` command will never give you a '+c' term.
If you need one, you must insert it yourself.



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$$\frac{d}{dx} \log(x^3 + x^2 + x + 1) = \frac{\frac{d}{dx}(x^3 + x^2 + x + 1)}{x^3 + x^2 + x + 1} = \frac{3x^2 + 2x + 1}{x^3 + x^2 + x + 1}.$$

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$$\begin{aligned} \log(x)^3 &= \frac{d}{dx} \left((a \log(x)^3 + b \log(x)^2 + c \log(x) + d)x \right) \\ &= (3a \log(x)^2 x^{-1} + 2b \log(x) x^{-1} + c x^{-1})x + \\ &\quad (a \log(x)^3 + b \log(x)^2 + c \log(x) + d). \end{aligned}$$

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$$\int \log(x)^3 dx = (a \log(x)^3 + b \log(x)^2 + c \log(x) + d)x$$

(How could we know this? — see later)

- ▶ **Problem:** find a, b, c and d .

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- ▶ So $a = 1, b + 3a = 0, c + 2b = 0$ and $d + c = 0$ (compare coefficients)

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- ▶ So $a = 1, b = -3, c = 6$ and $d = -6$

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$$\int \log(x)^3 dx = (\log(x)^3 - 3 \log(x)^2 + 6 \log(x) - 6)x.$$

$$\exp'(x) = \exp(x)$$

$$\sinh'(x) = \cosh(x)$$

$$\cosh'(x) = \sinh(x)$$

$$\tanh'(x) = \operatorname{sech}(x)^2$$

$$\sin'(x) = \cos(x)$$

$$\cos'(x) = -\sin(x)$$

$$\tan'(x) = \sec(x)^2$$

$$\log'(x) = 1/x$$

$$\operatorname{arcsinh}'(x) = (1 + x^2)^{-1/2}$$

$$\operatorname{arccosh}'(x) = (x^2 - 1)^{-1/2}$$

$$\operatorname{arctanh}'(x) = (1 - x^2)^{-1}$$

$$\operatorname{arcsin}'(x) = (1 - x^2)^{-1/2}$$

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$$\begin{aligned}\int \exp(x) dx &= \exp(x) \\ \int \cosh(x) dx &= \sinh(x) \\ \int \sinh(x) dx &= \cosh(x) \\ \int \operatorname{sech}(x)^2 dx &= \tanh(x) \\ \int \cos(x) dx &= \sin(x) \\ \int \sin(x) dx &= -\cos(x) \\ \int \sec(x)^2 dx &= \tan(x)\end{aligned}$$

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Rational functions

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▶ $\int \frac{4x^3 + 8}{x^6 - x^2} dx = \frac{8}{x} + 3 \ln(|x-1|) - \ln(|x+1|) - \ln(x^2+1) + 4 \arctan(x)$

Rational function examples

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- ▶ $\int \frac{x^2 + 1}{x^2 - 1} dx = x + \ln(|x - 1|) + \ln(|x + 1|)$
- ▶ $\int \left(\frac{x+1}{x-1}\right)^3 dx = 1 + \frac{6}{x-1} + \frac{12}{(x-1)^2} + \frac{8}{(x-1)^3}$
- ▶ $\int \frac{2x+2}{x^2+1} dx = \ln(x^2+1) + 2 \arctan(x)$
- ▶ $\int \frac{1}{x^{-1}+1+x} dx = \frac{1}{2} \ln(1+x+x^2) - \frac{1}{\sqrt{3}} \arctan\left(\frac{1+2x}{\sqrt{3}}\right)$
- ▶ $\int \frac{4}{1-x^4} dx = \ln(|x+1|) - \ln(|x-1|) + 2 \arctan(x)$

Rational function examples

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$$\frac{d}{dx} \ln(x^2 + ux + v) = \frac{2x + u}{x^2 + ux + v}$$

$$\frac{d}{dx} \arctan(ux + v) = \frac{u}{1 + (ux + v)^2} = \frac{u}{u^2x^2 + 2uvx + (v^2 + 1)}$$

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