Integration

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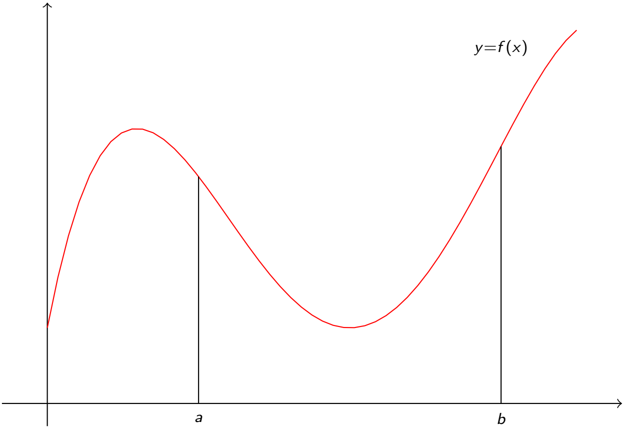
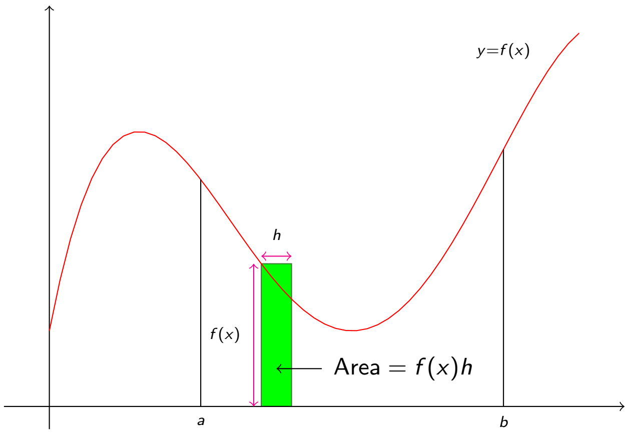
- If a particle moves with velocity $v(t) > 0$ at time t, then the total distance moved between times a and b is $\int_a^b v(t)\,dt$.
- A current flowing in a wire exerts a magnetic force on a moving electron. There is a formula for the force contributed by a short section of wire; to get the total force, we integrate.

Areas



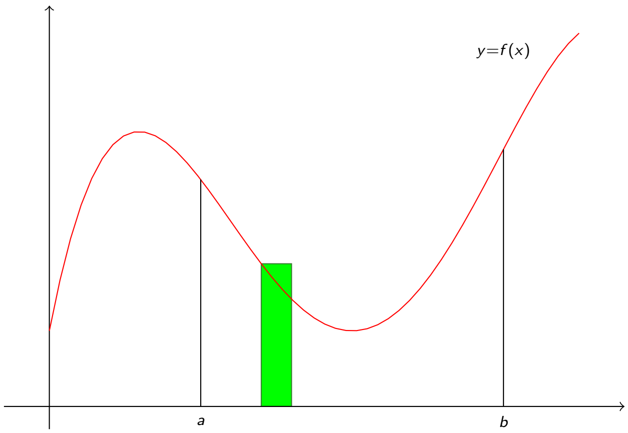
Consider the integral $\int_a^b f(x)\,dx$.

Areas



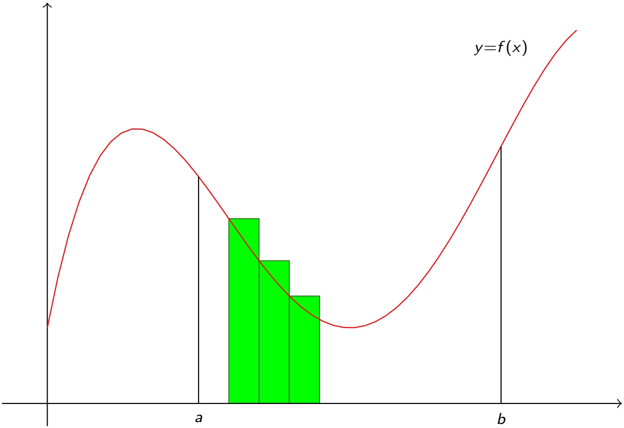
For each short interval $[x, x+h] \subset [a, b]$, we have a contribution $f(x)h$. This is the area of the green rectangle.

Areas



This is the contribution from one short interval, but we need to add together the contributions from many short intervals.

Areas



Here we have added in two more intervals

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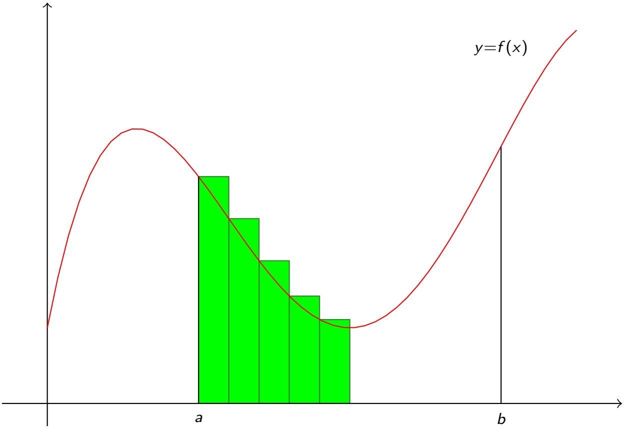

Here we have added in two more intervals – and two more

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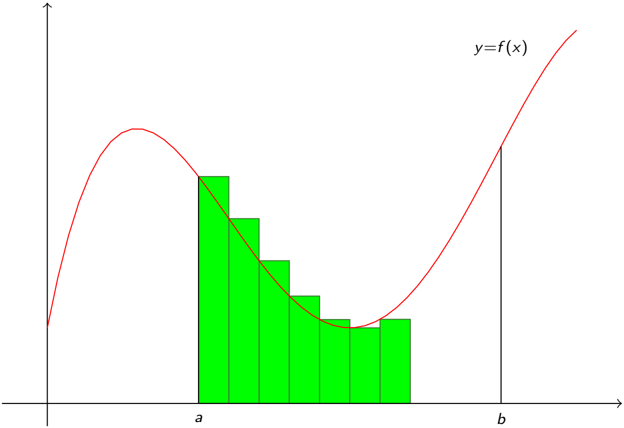

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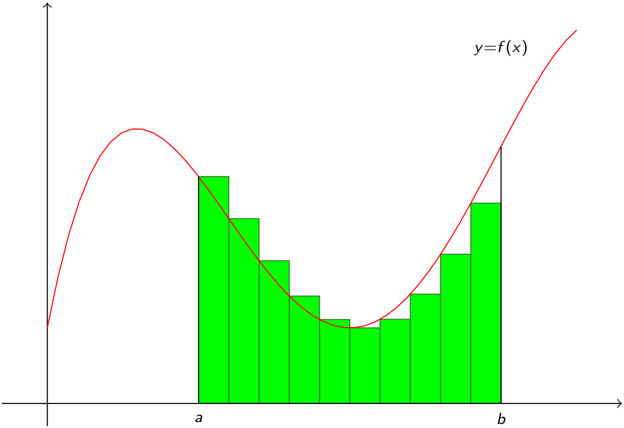
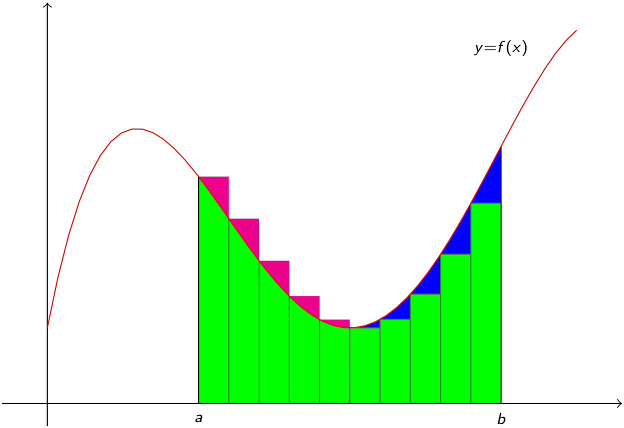


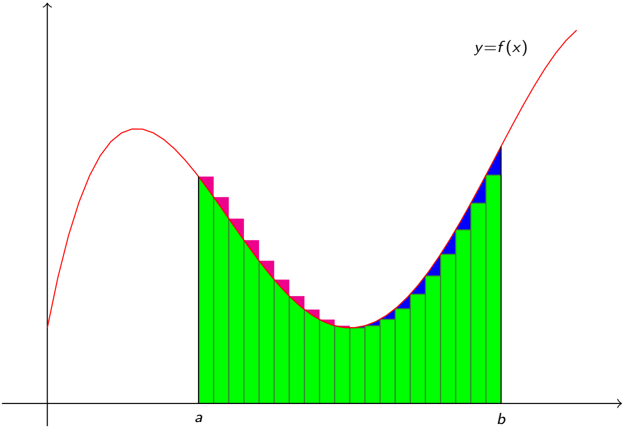
Now we have divided the whole interval $[a, b]$ into subintervals of length h. The sum of the terms $f(x)h$ is the area of the green region.

Areas



This is not exactly the same as the area under the curve, because of the regions marked in blue and pink.

Areas



However, the error decreases if we make h smaller.

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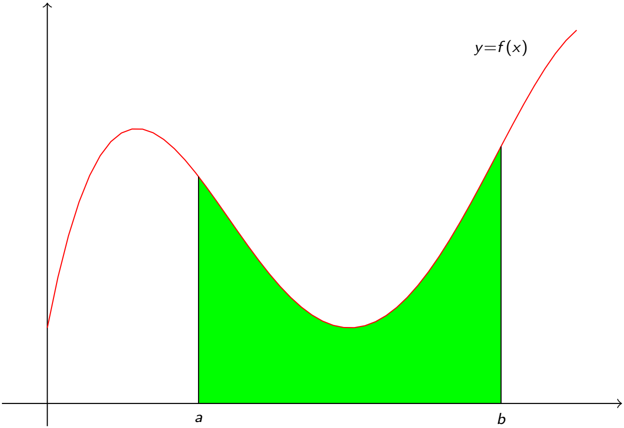

However, the error decreases if we make h smaller, and tends to zero in the limit.

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- An *indefinite integral* of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

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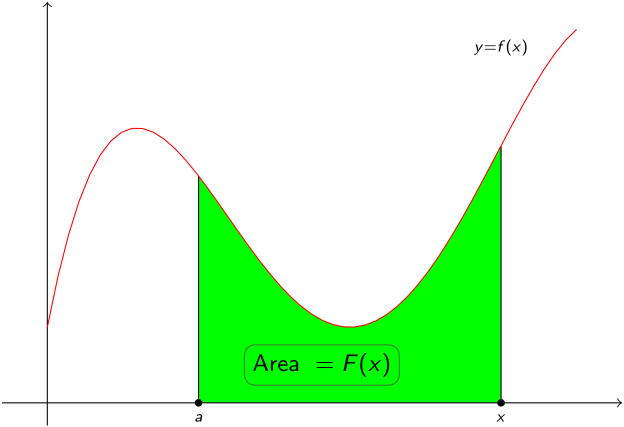
- $\int_a^b \frac{1}{x} = \left[\log(x)\right]_a^b = \log(b) - \log(a)$

Proof of the Fundamental Theorem



Choose a number a, and define $F(x) = \int_a^x f(t)\,dt$. We must show that $F'(x) = f(x)$.

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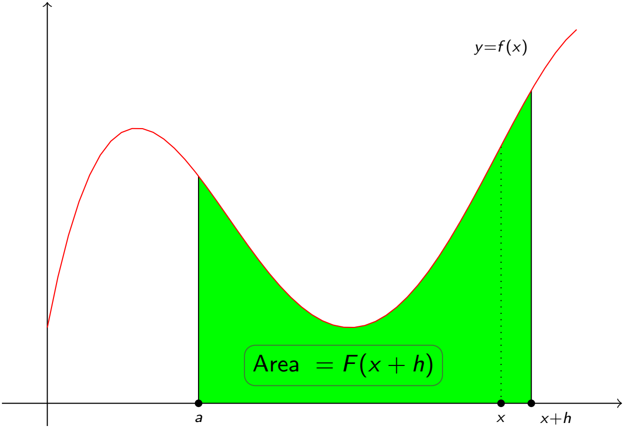

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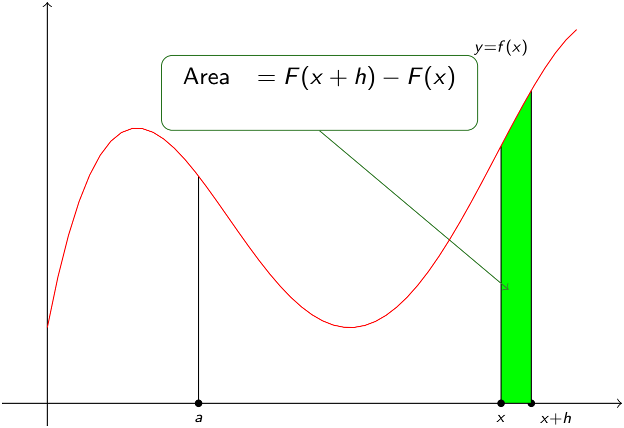

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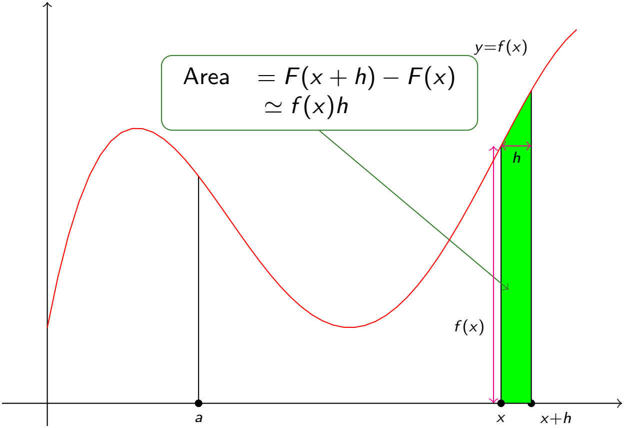

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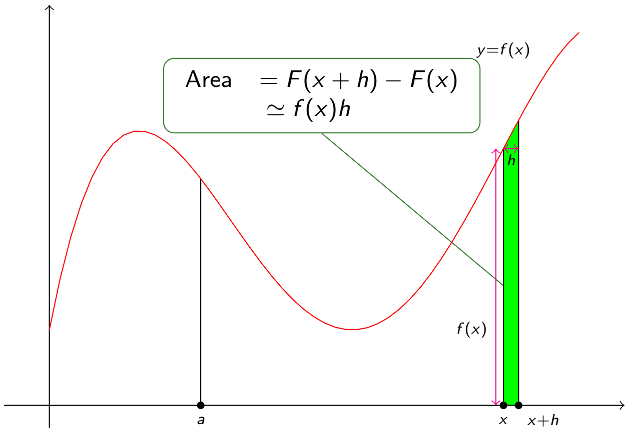

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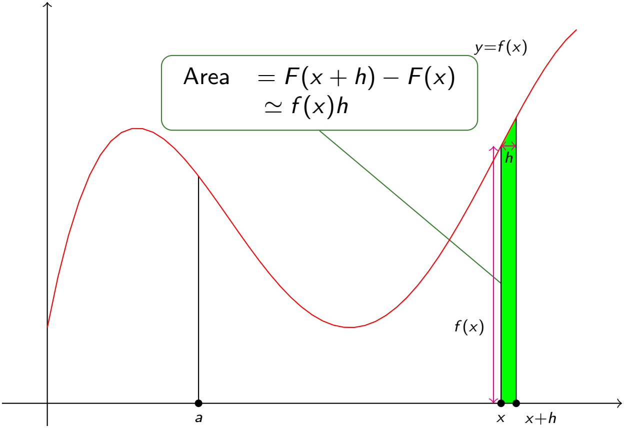

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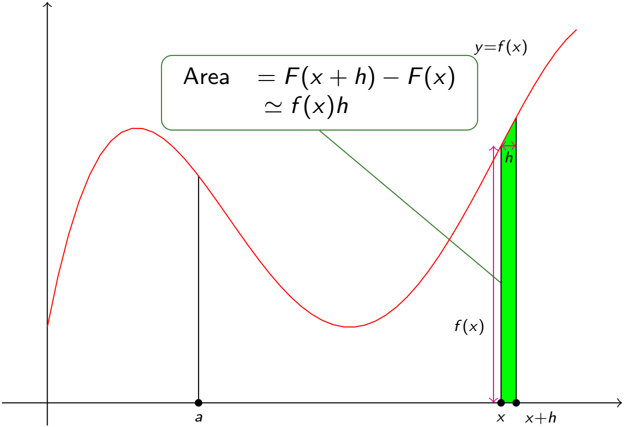

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$$F'(x) = \lim_{h \longrightarrow 0} (F(x+h) - F(x))/h = f(x).$$

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- Maple's `int()` command will never give you a '+c' term.
  If you need one, you must insert it yourself.

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- $\int \sin(x)^2\, dx \neq \sin(x)^3/3$, because

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- $\int 2x\, e^{x^2}\, dx = e^{x^2}$

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- $\int 2x\, e^{x^2} \, dx = e^{x^2}$, because $\frac{d}{dx} e^{x^2} = 2x\, e^{x^2}$.
- $\int \frac{3x^2+2x+1}{x^3+x^2+x+1} \, dx = \log(x^3 + x^2 + x + 1)$

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- $\int \frac{3x^2+2x+1}{x^3+x^2+x+1}\,dx = \log(x^3+x^2+x+1)$, because
$$\frac{d}{dx}\log(x^3+x^2+x+1) = \frac{\frac{d}{dx}(x^3+x^2+x+1)}{x^3+x^2+x+1} = \frac{3x^2+2x+1}{x^3+x^2+x+1}.$$

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  (How could we know this? — see later)
- **Problem:** find a, b, c and d.

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- Suppose we know that for some constants $a, \ldots, d$
$$\int \log(x)^3 \, dx = (a \log(x)^3 + b \log(x)^2 + c \log(x) + d)x$$
(How could we know this? — see later)
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- $$\begin{aligned} \log(x)^3 &= \frac{d}{dx}\Big((a\log(x)^3 + b\log(x)^2 + c\log(x) + d)x\Big) \\ &= (3a\log(x)^2x^{-1} + 2b\log(x)x^{-1} + cx^{-1})x + \\ &\quad (a\log(x)^3 + b\log(x)^2 + c\log(x) + d).1 \end{aligned}$$

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$$\int \log(x)^3 \, dx = (\log(x)^3 - 3 \log(x)^2 + 6 \log(x) - 6)x.$$

Standard integrals

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\begin{aligned}
\exp'(x) &= \exp(x) \\
\sinh'(x) &= \cosh(x) \\
\cosh'(x) &= \sinh(x) \\
\tanh'(x) &= \operatorname{sech}(x)^2 \\
\sin'(x) &= \cos(x) \\
\cos'(x) &= -\sin(x) \\
\tan'(x) &= \sec(x)^2
\end{aligned}
$$

$$
\begin{aligned}
\log'(x) &= 1/x \\
\operatorname{arcsinh}'(x) &= (1+x^2)^{-1/2} \\
\operatorname{arccosh}'(x) &= (x^2-1)^{-1/2} \\
\operatorname{arctanh}'(x) &= (1-x^2)^{-1} \\
\arcsin'(x) &= (1-x^2)^{-1/2} \\
\arccos'(x) &= -(1-x^2)^{-1/2} \\
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- If $f(x)$ is a rational function, then $\int f(x)\,dx$ is a sum of terms of the following types:
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  - Terms of the form $\arctan(ux+v)$.
- $\displaystyle\int \frac{4x^3+8}{x^6-x^2}\,dx = \frac{8}{x} + 3\ln(|x-1|) - \ln(|x+1|) - \ln(x^2+1) + 4\arctan(x)$

Rational function examples

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- $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx = x + \ln(|x-1|) + \ln(|x+1|)$
- $\displaystyle\int \left(\frac{x+1}{x-1}\right)^3 dx = 1 + \frac{6}{x-1} + \frac{12}{(x-1)^2} + \frac{8}{(x-1)^3}$
- $\displaystyle\int \frac{2x+2}{x^2+1}\,dx = \ln(x^2+1) + 2\arctan(x)$
- $\displaystyle\int \frac{1}{x^{-1}+1+x}\,dx = \frac{1}{2}\ln(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{1+2x}{\sqrt{3}}\right)$
- $\displaystyle\int \frac{4}{1-x^4}\,dx = \ln(|x+1|) - \ln(|x-1|) + 2\arctan(x)$

Rational function examples

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- $$\int \frac{1}{x^{-1}+1+x}\,dx = \frac{1}{2}\ln(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{1+2x}{\sqrt{3}}\right)$$
- $$\int \frac{4}{1-x^4}\,dx = \ln(|x+1|) - \ln(|x-1|) + 2\arctan(x)$$

$$\frac{d}{dx}\ln(|x-u|) = \frac{1}{x-u} \qquad \frac{d}{dx}\ln(x^2+ux+v) = \frac{2x+u}{x^2+ux+v}$$

$$\frac{d}{dx}\arctan(ux+v) = \frac{u}{1+(ux+v)^2} = \frac{u}{u^2x^2+2uvx+(v^2+1)}$$

Trigonometric polynomials

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$$\begin{aligned}\cos(2x) &= \cos(x)^2 - \sin(x)^2 = 2\cos(x)^2 - 1 = 1 - 2\sin(x)^2 \\ \sin(x)^2 &= 1/2 - \cos(2x)/2 \\ \int \sin(x)^2\,dx &= x/2 - \sin(2x)/4\end{aligned}$$

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\int \sin(x)^3\,dx &= -3\cos(x)/4 + \cos(3x)/12
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\sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4
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\int \sin(x)\sin(2x)\sin(4x)\,dx &= \cos(x)/4 - \cos(3x)/12 - \cos(5x)/20 + \cos(7x)/28
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\int \sin(x)^3\,dx &= -3\cos(x)/4 + \cos(3x)/12 \\
\sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\
\int \sin(x)\sin(2x)\sin(4x)\,dx &= \cos(x)/4 - \cos(3x)/12 - \cos(5x)/20 + \cos(7x)/28 \\
\sin(x)^4 + \cos(x)^4 &= 3/4 + \cos(4x)/4
\end{aligned}$$

Trigonometric polynomials

$$\boxed{\int \sin(nx)\,dx = -\cos(nx)/n \qquad\qquad \int \cos(nx)\,dx = \sin(nx)/n}$$

$$\begin{aligned}
\cos(2x) &= \cos(x)^2 - \sin(x)^2 = 2\cos(x)^2 - 1 = 1 - 2\sin(x)^2 \\
\sin(x)^2 &= 1/2 - \cos(2x)/2 \\
\int \sin(x)^2\,dx &= x/2 - \sin(2x)/4 \\
\int \cos(x)^2\,dx &= x/2 + \sin(2x)/4 \\
\sin(x)^3 &= 3\sin(x)/4 - \sin(3x)/4 \\
\int \sin(x)^3\,dx &= -3\cos(x)/4 + \cos(3x)/12 \\
\sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\
\int \sin(x)\sin(2x)\sin(4x)\,dx &= \cos(x)/4 - \cos(3x)/12 - \cos(5x)/20 + \cos(7x)/28 \\
\sin(x)^4 + \cos(x)^4 &= 3/4 + \cos(4x)/4 \\
\int \sin(x)^4 + \cos(x)^4\,dx &= 3x/4 + \sin(4x)/16
\end{aligned}$$