Integration 2

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Exponential oscillations

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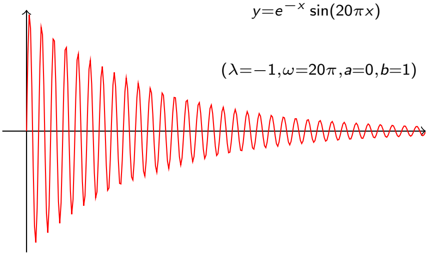

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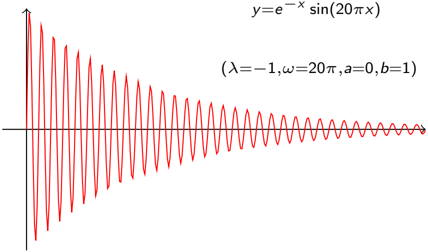

- Special cases:

$$f(x) = e^{\lambda x}\sin(\omega x) \qquad (a = 0, b = 1)$$
$$f(x) = a\cos(\omega x) + b\sin(\omega x) \qquad (\lambda = 0)$$
$$f(x) = ae^{\lambda x} \qquad (\omega = 0).$$

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$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(\cos(4x) + 13\sin(4x))/10$$

Integrating exponential oscillations

Alternatively:

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(A\cos(4x) + B\sin(4x)) \text{ for some } A, B$$

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By comparing coefficients, we must have $4B - 2A = 5$ and $2B + 4A = 3$. These equations can be solved to give $A = 1/10$ and $B = 13/10$. Thus

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(\cos(4x) + 13\sin(4x))/10.$$

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- A *polynomial exponential oscillation* is a function of the form

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- The function $f(x) = e^{-2x}((1 + x^5)\cos(4x) + x^3\sin(4x))$ is a PEO of growth rate -2, frequency 4 and degree 5.

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- The function $f(x) = e^{4x}((1 + x^3 + x^6)\sin(3x))$ is a PEO of growth rate 4, frequency 3 and degree 6.
- **Fact:** The integral of any PEO is another PEO with the same growth rate, frequency and degree.

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  for some A, B, C, D.
- $$\begin{aligned} xe^{-x}\sin(x) &= \tfrac{d}{dx}\left((Ax+B)e^{-x}\cos(x) + (Cx+D)e^{-x}\sin(x)\right) \\ &= Ae^{-x}\cos(x) - (Ax+B)e^{-x}\cos(x) - (Ax+B)e^{-x}\sin(x) + \\ &\quad Ce^{-x}\sin(x) - (Cx+D)e^{-x}\sin(x) + (Cx+D)e^{-x}\cos(x) \end{aligned}$$

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- So $A=-1/2$, $B=-1/2$, $C=-1/2$, $D=0$

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- $\int xe^{-x}\sin(x)\,dx = -((x+1)e^{-x}\cos(x) + xe^{-x}\sin(x))/2$.

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- $$\begin{aligned} x^3 e^x &= \tfrac{d}{dx}\left((Ax^3 + Bx^2 + Cx + D)e^x\right) \\ &= (3Ax^2 + 2Bx + C)e^x + (Ax^3 + Bx^2 + Cx + D)e^x \end{aligned}$$

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- $A = 1$, $3A + B = 0$, $2B + C = 0$, $C + D = 0$.
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- $\int x^3 e^x \, dx$ is a PEO of degree 3, growth 1 and frequency 0.
- $\int x^3 e^x \, dx = (Ax^3 + Bx^2 + Cx + D)e^x$ for some A, B, C, D.
- $$\begin{aligned} x^3 e^x &= \tfrac{d}{dx}\Big((Ax^3 + Bx^2 + Cx + D)e^x\Big) \\ &= (3Ax^2 + 2Bx + C)e^x + (Ax^3 + Bx^2 + Cx + D)e^x \\ &= (Ax^3 + (3A + B)x^2 + (2B + C)x + (C + D))e^x. \end{aligned}$$
- $A = 1$, $3A + B = 0$, $2B + C = 0$, $C + D = 0$.
- so $A = 1$, $B = -3$, $C = 6$, $D = -6$
- so $\int x^3 e^x \, dx = (x^3 - 3x^2 + 6x - 6)e^x$.

Integration by parts — I

Integration by parts — I

- Consider $\int xe^{x/a}\,dx$.

- Consider $\int x e^{x/a}\,dx$.
- $u = x$ $\qquad$ $dv/dx = e^{x/a}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.

Integration by parts — I

- Consider $\int xe^{x/a}\,dx$.
- $u = x$ $\qquad$ $dv/dx = e^{x/a}$
- $du/dx = 1$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.

Integration by parts — I

- Consider $\int x e^{x/a}\, dx$.
- $u = x$ $\qquad$ $dv/dx = e^{x/a}$
- $du/dx = 1$ $\qquad$ $v = a\, e^{x/a}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.

Integration by parts — I

- Consider $\int xe^{x/a}\, dx$.
- $u = x$ $\qquad$ $dv/dx = e^{x/a}$
- $du/dx = 1$ $\qquad$ $v = a\, e^{x/a}$
- $\int xe^{x/a}\, dx = uv - \int \frac{du}{dx} v\, dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx}$$

Integration by parts — I

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- $u = x$ $\qquad$ $dv/dx = e^{x/a}$
- $du/dx = 1$ $\qquad$ $v = a\, e^{x/a}$
- $\int x e^{x/a}\, dx = uv - \int \frac{du}{dx} v\, dx = axe^{x/a} - \int a\, e^{x/a}\, dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
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- This is most useful when (a) du/dx is simpler than u (eg u polynomial) and (b) v is no more complicated than dv/dx (eg $dv/dx = \cos(x)$).

Integration by parts — II

Integration by parts — II

- Consider $\int (1 - \ln(x))x^{-2}\,dx$.

Integration by parts — II

- Consider $\int (1 - \ln(x))x^{-2}\, dx$.
- $u = 1 - \ln(x)$ $dv/dx = x^{-2}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.

Integration by parts — II

- Consider $\int (1 - \ln(x))x^{-2}\,dx$.
- $u = 1 - \ln(x)$ $\qquad dv/dx = x^{-2}$
- $du/dx = -x^{-1}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.

Integration by parts — II

- Consider $\int (1 - \ln(x))x^{-2}\, dx$.
- $u = 1 - \ln(x)$ $\qquad dv/dx = x^{-2}$
- $du/dx = -x^{-1}$ $\qquad v = -x^{-1}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.

Integration by parts — II

- Consider $\int (1 - \ln(x))x^{-2}\,dx$.
- $u = 1 - \ln(x)$ $\quad$ $dv/dx = x^{-2}$
- $du/dx = -x^{-1}$ $\quad$ $v = -x^{-1}$
- $\int (1 - \ln(x))x^{-2}\,dx = uv - \int \frac{du}{dx} v\,dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\int u \frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx$$

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- Consider $\int (1 - \ln(x))x^{-2}\, dx$.
- $u = 1 - \ln(x)$ $\qquad$ $dv/dx = x^{-2}$
- $du/dx = -x^{-1}$ $\qquad$ $v = -x^{-1}$
- $\int (1 - \ln(x))x^{-2}\, dx = uv - \int \frac{du}{dx} v\, dx = -(1 - \ln(x))x^{-1} - \int x^{-2}\, dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx$$

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- $u = 1 - \ln(x)$ $\qquad$ $dv/dx = x^{-2}$
- $du/dx = -x^{-1}$ $\qquad$ $v = -x^{-1}$
- $\int (1 - \ln(x))x^{-2}\, dx = uv - \int \frac{du}{dx} v\, dx = -(1 - \ln(x))x^{-1} - \int x^{-2}\, dx$

  $= (\ln(x) - 1)x^{-1} + x^{-1}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx$$

Integration by parts — II

- Consider $\int (1 - \ln(x))x^{-2}\, dx$.
- $u = 1 - \ln(x)$ $\qquad dv/dx = x^{-2}$
- $du/dx = -x^{-1}$ $\qquad v = -x^{-1}$
- $\int (1 - \ln(x))x^{-2}\, dx = uv - \int \frac{du}{dx} v\, dx = -(1 - \ln(x))x^{-1} - \int x^{-2}\, dx$

  $= (\ln(x) - 1)x^{-1} + x^{-1} = \ln(x)/x$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx$$

Integration by parts — III

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ $\qquad dv/dx = \sin(\omega x)$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ $dv/dx = \sin(\omega x)$
- $du/dx = 1$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ — $dv/dx = \sin(\omega x)$
- $du/dx = 1$ — $v = -\omega^{-1} \cos(\omega x)$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ $\qquad dv/dx = \sin(\omega x)$
- $du/dx = 1$ $\qquad v = -\omega^{-1}\cos(\omega x)$
- $\int x \sin(\omega x)\, dx = uv - \int \frac{du}{dx} v\, dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx}$$

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ $\qquad$ $dv/dx = \sin(\omega x)$
- $du/dx = 1$ $\qquad$ $v = -\omega^{-1}\cos(\omega x)$
- $\int x \sin(\omega x)\, dx = uv - \int \frac{du}{dx} v\, dx = -\omega^{-1} x \cos(\omega x) + \int \omega^{-1} \cos(\omega x)\, dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx}$$

Integration by parts — III

- Consider $\int x \sin(\omega x)\, dx$.
- $u = x$ $\quad dv/dx = \sin(\omega x)$
- $du/dx = 1$ $\quad v = -\omega^{-1}\cos(\omega x)$
- $\int x \sin(\omega x)\, dx = uv - \int \frac{du}{dx} v\, dx = -\omega^{-1} x \cos(\omega x) + \int \omega^{-1} \cos(\omega x)\, dx$

  $= -\omega^{-1} x \cos(\omega x) + \omega^{-2} \sin(\omega x)$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx}$$

Integration by parts — IV

Integration by parts — IV

- Consider $\int \arcsin(x)\, dx$.

Integration by parts — IV

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.

Integration by parts — IV

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\quad$ $dv/dx = 1$
- $du/dx = (1 - x^2)^{-1/2}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.

Integration by parts — IV

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$
- $du/dx = (1 - x^2)^{-1/2}$ $\qquad v = x$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$
- $du/dx = (1 - x^2)^{-1/2}$ $\qquad v = x$
- $\int \arcsin(x).1\,dx = uv - \int \frac{du}{dx} v\,dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u \frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx}$$

Integration by parts — IV

- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$
- $du/dx = (1 - x^2)^{-1/2}$ $\qquad v = x$
- $\int \arcsin(x).1\,dx = uv - \int \frac{du}{dx} v\,dx = \arcsin(x).x - \int x(1 - x^2)^{-1/2}\,dx$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx}$$

Integration by parts — IV

- Consider $\int \arcsin(x).1\, dx$.
- $u = \arcsin(x)$ $\qquad dv/dx = 1$
- $du/dx = (1 - x^2)^{-1/2}$ $\qquad v = x$
- $\int \arcsin(x).1\, dx = uv - \int \frac{du}{dx} v\, dx = \arcsin(x).x - \int x(1-x^2)^{-1/2}\, dx$

  $= x \arcsin(x) + (1 - x^2)^{1/2}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx}$$