Taylor Series

Introduction

Introduction

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Introduction

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{k=0}^{\infty} kx^k \qquad (\text{ for } |x| < 1)$$

Introduction

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{k=0}^{\infty} kx^k \qquad (\text{ for } |x| < 1)$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Introduction

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{k=0}^{\infty} kx^k \qquad (\text{ for } |x| < 1)$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

Introduction

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{k=0}^{\infty} kx^k \qquad (\text{ for } |x| < 1)$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

- For any reasonable function $f(x)$, there are coefficients a_k such that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

(when x is sufficiently small). This is the *Taylor series* for $f(x)$.

Exceptions

Exceptions

Not every function has a Taylor series.

Exceptions

Not every function has a Taylor series.

- $f_0(x) = 1/x$ does not, because $f_0(0)$ is undefined.

Exceptions

Not every function has a Taylor series.

- $f_0(x) = 1/x$ does not, because $f_0(0)$ is undefined.
- $f_1(x) = |x|$ and $f_2(x) = x^{1/3}$ do not, because the slopes $f_1'(0)$ and $f_2'(0)$ are not defined.

Exceptions

Not every function has a Taylor series.

- $f_0(x) = 1/x$ does not, because $f_0(0)$ is undefined.
- $f_1(x) = |x|$ and $f_2(x) = x^{1/3}$ do not, because the slopes $f_1'(0)$ and $f_2'(0)$ are not defined.
- $f_3(x) = \ln(x)$ does not, because $f_3'(x)$ is undefined for $x < 0$.

Exceptions

Not every function has a Taylor series.

- $f_0(x) = 1/x$ does not, because $f_0(0)$ is undefined.
- $f_1(x) = |x|$ and $f_2(x) = x^{1/3}$ do not, because the slopes $f_1'(0)$ and $f_2'(0)$ are not defined.
- $f_3(x) = \ln(x)$ does not, because $f_3'(x)$ is undefined for $x < 0$.
- $f_4(x) = e^{-1/x^2}$ does not, for a more subtle reason.

Exceptions

Not every function has a Taylor series.

- $f_0(x) = 1/x$ does not, because $f_0(0)$ is undefined.
- $f_1(x) = |x|$ and $f_2(x) = x^{1/3}$ do not, because the slopes $f_1'(0)$ and $f_2'(0)$ are not defined.
- $f_3(x) = \ln(x)$ does not, because $f_3'(x)$ is undefined for $x < 0$.
- $f_4(x) = e^{-1/x^2}$ does not, for a more subtle reason.

For a full explanation, see Level 3 complex analysis.

Truncated series

Truncated series

Often we only calculate with finitely many terms of the Taylor series.

Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

The notation $O(x^7)$ means that there are extra terms involving powers x^k with $k \geq 7$.

Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

The notation $O(x^7)$ means that there are extra terms involving powers x^k with $k \geq 7$. The above is the *7th order Taylor series* for $\tan(x)$. It is a good approximation to $\tan(x)$ if x is sufficiently small.

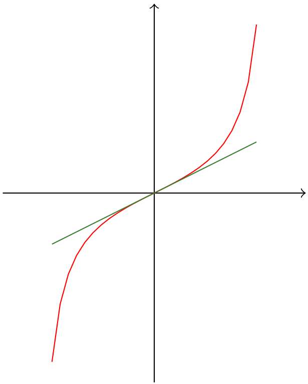
Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

The notation $O(x^7)$ means that there are extra terms involving powers x^k with $k \geq 7$. The above is the *7th order Taylor series* for $\tan(x)$. It is a good approximation to $\tan(x)$ if x is sufficiently small.

$\tan(x) = x + O(x^3)$



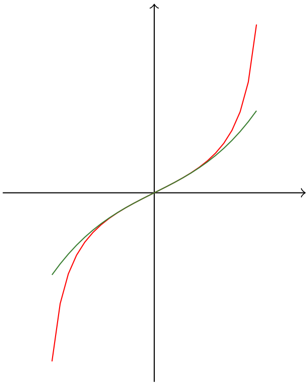
Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

The notation $O(x^7)$ means that there are extra terms involving powers x^k with $k \geq 7$. The above is the *7th order Taylor series* for $\tan(x)$. It is a good approximation to $\tan(x)$ if x is sufficiently small.

$\tan(x) = x + x^3/3 + O(x^5)$



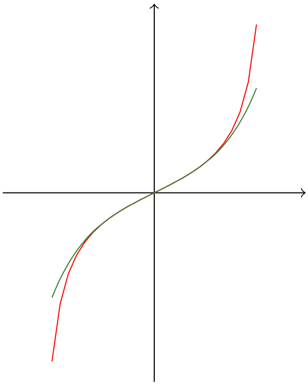
Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

The notation $O(x^7)$ means that there are extra terms involving powers x^k with $k \geq 7$. The above is the *7th order Taylor series* for $\tan(x)$. It is a good approximation to $\tan(x)$ if x is sufficiently small.

$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$



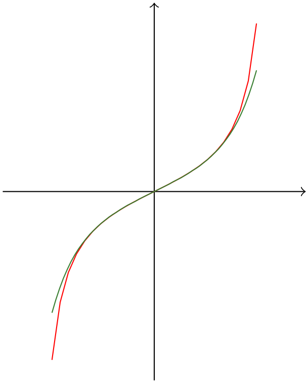
Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

The notation $O(x^7)$ means that there are extra terms involving powers x^k with $k \geq 7$. The above is the *7th order Taylor series* for $\tan(x)$. It is a good approximation to $\tan(x)$ if x is sufficiently small.

$\tan(x) = x + x^3/3 + 2x^5/15 + 17x^7/315 + O(x^9)$



Finding coefficients

Finding coefficients

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = \frac{1}{k!} \left. \frac{d^k y}{dx^k} \right|_{x=0}$$

Finding coefficients

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = \frac{1}{k!} \left. \frac{d^k y}{dx^k} \right|_{x=0}$$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = f^{(k)}(0)/k!$$

Finding coefficients

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = \frac{1}{k!} \left. \frac{d^k y}{dx^k} \right|_{x=0}$$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = f^{(k)}(0)/k!$$

Example:

$$\exp^{(k)}(x) = \cdots = \exp'''(x) = \exp''(x) = \exp'(x) = \exp(x) = e^x$$

Finding coefficients

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = \frac{1}{k!} \left. \frac{d^k y}{dx^k} \right|_{x=0}$$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = f^{(k)}(0)/k!$$

Example:

$$\exp^{(k)}(x) = \cdots = \exp'''(x) = \exp''(x) = \exp'(x) = \exp(x) = e^x$$

$$\exp^{(k)}(0) = \cdots = \exp'''(0) = \exp''(0) = \exp'(0) = \exp(0) = 1$$

Finding coefficients

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = \frac{1}{k!} \left. \frac{d^k y}{dx^k} \right|_{x=0}$$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = f^{(k)}(0)/k!$$

Example:

$$\exp^{(k)}(x) = \cdots = \exp'''(x) = \exp''(x) = \exp'(x) = \exp(x) = e^x$$

$$\exp^{(k)}(0) = \cdots = \exp'''(0) = \exp''(0) = \exp'(0) = \exp(0) = 1$$

Thus $a_k = 1/k!$, and $\exp(x) = \sum_k x^k/k!$.

Another example

Another example

Take $f(x) = \sin(x)$.

Another example

Take $f(x) = \sin(x)$.

$$f(x) = \sin(x) \qquad f'(x) = \cos(x) \qquad f''(x) = -\sin(x) \qquad f'''(x) = -\cos(x)$$

Another example

Take $f(x) = \sin(x)$.

$$\begin{array}{cccc} f(x) = \sin(x) & f'(x) = \cos(x) & f''(x) = -\sin(x) & f'''(x) = -\cos(x) \\ f^{(4)}(x) = \sin(x) & f^{(5)}(x) = \cos(x) & f^{(6)}(x) = -\sin(x) & f^{(7)}(x) = -\cos(x) \end{array}$$

Another example

Take $f(x) = \sin(x)$.

$$\begin{array}{rlrlrlrl}
f(x) &= \sin(x) & f'(x) &= \cos(x) & f''(x) &= -\sin(x) & f'''(x) &= -\cos(x) \\
f^{(4)}(x) &= \sin(x) & f^{(5)}(x) &= \cos(x) & f^{(6)}(x) &= -\sin(x) & f^{(7)}(x) &= -\cos(x) \\
f^{(8)}(x) &= \sin(x) & f^{(9)}(x) &= \cos(x) & f^{(10)}(x) &= -\sin(x) & f^{(11)}(x) &= -\cos(x)
\end{array}$$

Another example

Take $f(x) = \sin(x)$.

$$
\begin{array}{llll}
f(x) = \sin(x) & f'(x) = \cos(x) & f''(x) = -\sin(x) & f'''(x) = -\cos(x) \\
f^{(4)}(x) = \sin(x) & f^{(5)}(x) = \cos(x) & f^{(6)}(x) = -\sin(x) & f^{(7)}(x) = -\cos(x) \\
f^{(8)}(x) = \sin(x) & f^{(9)}(x) = \cos(x) & f^{(10)}(x) = -\sin(x) & f^{(11)}(x) = -\cos(x)
\end{array}
$$

$$
\begin{array}{llll}
f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f'''(0) = -1
\end{array}
$$

Another example

Take $f(x) = \sin(x)$.

$$
\begin{array}{llll}
f(x) = \sin(x) & f'(x) = \cos(x) & f''(x) = -\sin(x) & f'''(x) = -\cos(x) \\
f^{(4)}(x) = \sin(x) & f^{(5)}(x) = \cos(x) & f^{(6)}(x) = -\sin(x) & f^{(7)}(x) = -\cos(x) \\
f^{(8)}(x) = \sin(x) & f^{(9)}(x) = \cos(x) & f^{(10)}(x) = -\sin(x) & f^{(11)}(x) = -\cos(x)
\end{array}
$$

$$
\begin{array}{llll}
f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f'''(0) = -1 \\
f^{(4)}(0) = 0 & f^{(5)}(0) = 1 & f^{(6)}(0) = 0 & f^{(7)}(0) = -1
\end{array}
$$

Another example

Take $f(x) = \sin(x)$.

$$\begin{array}{rlrlrlrl}
f(x) &= \sin(x) & f'(x) &= \cos(x) & f''(x) &= -\sin(x) & f'''(x) &= -\cos(x) \\
f^{(4)}(x) &= \sin(x) & f^{(5)}(x) &= \cos(x) & f^{(6)}(x) &= -\sin(x) & f^{(7)}(x) &= -\cos(x) \\
f^{(8)}(x) &= \sin(x) & f^{(9)}(x) &= \cos(x) & f^{(10)}(x) &= -\sin(x) & f^{(11)}(x) &= -\cos(x)
\end{array}$$

$$\begin{array}{rlrlrlrl}
f(0) &= 0 & f'(0) &= 1 & f''(0) &= 0 & f'''(0) &= -1 \\
f^{(4)}(0) &= 0 & f^{(5)}(0) &= 1 & f^{(6)}(0) &= 0 & f^{(7)}(0) &= -1 \\
f^{(8)}(0) &= 0 & f^{(9)}(0) &= 1 & f^{(10)}(0) &= 0 & f^{(11)}(0) &= -1
\end{array}$$

Another example

Take $f(x) = \sin(x)$.

$$
\begin{aligned}
f(x) &= \sin(x) & f'(x) &= \cos(x) & f''(x) &= -\sin(x) & f'''(x) &= -\cos(x) \\
f^{(4)}(x) &= \sin(x) & f^{(5)}(x) &= \cos(x) & f^{(6)}(x) &= -\sin(x) & f^{(7)}(x) &= -\cos(x) \\
f^{(8)}(x) &= \sin(x) & f^{(9)}(x) &= \cos(x) & f^{(10)}(x) &= -\sin(x) & f^{(11)}(x) &= -\cos(x)
\end{aligned}
$$

$$
\begin{aligned}
f(0) &= 0 & f'(0) &= 1 & f''(0) &= 0 & f'''(0) &= -1 \\
f^{(4)}(0) &= 0 & f^{(5)}(0) &= 1 & f^{(6)}(0) &= 0 & f^{(7)}(0) &= -1 \\
f^{(8)}(0) &= 0 & f^{(9)}(0) &= 1 & f^{(10)}(0) &= 0 & f^{(11)}(0) &= -1
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= 0 & a_1 &= 1 & a_2 &= 0 & a_3 &= -1/3!
\end{aligned}
$$

Another example

Take $f(x) = \sin(x)$.

$$
\begin{aligned}
f(x) &= \sin(x) & f'(x) &= \cos(x) & f''(x) &= -\sin(x) & f'''(x) &= -\cos(x) \\
f^{(4)}(x) &= \sin(x) & f^{(5)}(x) &= \cos(x) & f^{(6)}(x) &= -\sin(x) & f^{(7)}(x) &= -\cos(x) \\
f^{(8)}(x) &= \sin(x) & f^{(9)}(x) &= \cos(x) & f^{(10)}(x) &= -\sin(x) & f^{(11)}(x) &= -\cos(x)
\end{aligned}
$$

$$
\begin{aligned}
f(0) &= 0 & f'(0) &= 1 & f''(0) &= 0 & f'''(0) &= -1 \\
f^{(4)}(0) &= 0 & f^{(5)}(0) &= 1 & f^{(6)}(0) &= 0 & f^{(7)}(0) &= -1 \\
f^{(8)}(0) &= 0 & f^{(9)}(0) &= 1 & f^{(10)}(0) &= 0 & f^{(11)}(0) &= -1
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= 0 & a_1 &= 1 & a_2 &= 0 & a_3 &= -1/3! \\
a_4 &= 0 & a_5 &= 1/5! & a_6 &= 0 & a_7 &= -1/7!
\end{aligned}
$$

Another example

Take $f(x) = \sin(x)$.

$$
\begin{array}{llll}
f(x) = \sin(x) & f'(x) = \cos(x) & f''(x) = -\sin(x) & f'''(x) = -\cos(x) \\
f^{(4)}(x) = \sin(x) & f^{(5)}(x) = \cos(x) & f^{(6)}(x) = -\sin(x) & f^{(7)}(x) = -\cos(x) \\
f^{(8)}(x) = \sin(x) & f^{(9)}(x) = \cos(x) & f^{(10)}(x) = -\sin(x) & f^{(11)}(x) = -\cos(x)
\end{array}
$$

$$
\begin{array}{llll}
f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f'''(0) = -1 \\
f^{(4)}(0) = 0 & f^{(5)}(0) = 1 & f^{(6)}(0) = 0 & f^{(7)}(0) = -1 \\
f^{(8)}(0) = 0 & f^{(9)}(0) = 1 & f^{(10)}(0) = 0 & f^{(11)}(0) = -1
\end{array}
$$

$$
\begin{array}{llll}
a_0 = 0 & a_1 = 1 & a_2 = 0 & a_3 = -1/3! \\
a_4 = 0 & a_5 = 1/5! & a_6 = 0 & a_7 = -1/7! \\
a_8 = 0 & a_9 = 1/9! & a_{10} = 0 & a_{11} = -1/11!
\end{array}
$$

Another example

Take $f(x) = \sin(x)$.

$$\begin{aligned} f(x) &= \sin(x) & f'(x) &= \cos(x) & f''(x) &= -\sin(x) & f'''(x) &= -\cos(x) \\ f^{(4)}(x) &= \sin(x) & f^{(5)}(x) &= \cos(x) & f^{(6)}(x) &= -\sin(x) & f^{(7)}(x) &= -\cos(x) \\ f^{(8)}(x) &= \sin(x) & f^{(9)}(x) &= \cos(x) & f^{(10)}(x) &= -\sin(x) & f^{(11)}(x) &= -\cos(x) \end{aligned}$$

$$\begin{aligned} f(0) &= 0 & f'(0) &= 1 & f''(0) &= 0 & f'''(0) &= -1 \\ f^{(4)}(0) &= 0 & f^{(5)}(0) &= 1 & f^{(6)}(0) &= 0 & f^{(7)}(0) &= -1 \\ f^{(8)}(0) &= 0 & f^{(9)}(0) &= 1 & f^{(10)}(0) &= 0 & f^{(11)}(0) &= -1 \end{aligned}$$

$$\begin{aligned} a_0 &= 0 & a_1 &= 1 & a_2 &= 0 & a_3 &= -1/3! \\ a_4 &= 0 & a_5 &= 1/5! & a_6 &= 0 & a_7 &= -1/7! \\ a_8 &= 0 & a_9 &= 1/9! & a_{10} &= 0 & a_{11} &= -1/11! \end{aligned}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

Another example

Take $f(x) = \sin(x)$.

$$\begin{aligned} f(x) &= \sin(x) & f'(x) &= \cos(x) & f''(x) &= -\sin(x) & f'''(x) &= -\cos(x) \\ f^{(4)}(x) &= \sin(x) & f^{(5)}(x) &= \cos(x) & f^{(6)}(x) &= -\sin(x) & f^{(7)}(x) &= -\cos(x) \\ f^{(8)}(x) &= \sin(x) & f^{(9)}(x) &= \cos(x) & f^{(10)}(x) &= -\sin(x) & f^{(11)}(x) &= -\cos(x) \end{aligned}$$

$$\begin{aligned} f(0) &= 0 & f'(0) &= 1 & f''(0) &= 0 & f'''(0) &= -1 \\ f^{(4)}(0) &= 0 & f^{(5)}(0) &= 1 & f^{(6)}(0) &= 0 & f^{(7)}(0) &= -1 \\ f^{(8)}(0) &= 0 & f^{(9)}(0) &= 1 & f^{(10)}(0) &= 0 & f^{(11)}(0) &= -1 \end{aligned}$$

$$\begin{aligned} a_0 &= 0 & a_1 &= 1 & a_2 &= 0 & a_3 &= -1/3! \\ a_4 &= 0 & a_5 &= 1/5! & a_6 &= 0 & a_7 &= -1/7! \\ a_8 &= 0 & a_9 &= 1/9! & a_{10} &= 0 & a_{11} &= -1/11! \end{aligned}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Other methods

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$\cosh(x)$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\,(k!)}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\,(k!)} = \sum_{k \text{even}} \frac{x^k}{k!}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$\sinh(x)/x$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x)$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\,(k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\,(k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x)$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\,(k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\,(k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x) = 1 + x + x^2 + x^3 + \ldots$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\,(k!)} = \sum_{k\text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\,(k!)} = \sum_{k\text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x) = 1 + x + x^2 + x^3 + \dots = \sum_k x^k$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\,(k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\,(k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x) = 1 + x + x^2 + x^3 + \ldots = \sum_k x^k$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x) = 1 + x + x^2 + x^3 + \ldots = \sum_k x^k$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x \frac{d}{dx} \sum_k x^k$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k\text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)} = \sum_{k\text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x) = 1 + x + x^2 + x^3 + \ldots = \sum_k x^k$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x \frac{d}{dx} \sum_k x^k = x \sum_k k\, x^{k-1}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x) = 1 + x + x^2 + x^3 + \ldots = \sum_k x^k$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x \frac{d}{dx} \sum_k x^k = x \sum_k k\, x^{k-1} = \sum_k k\, x^k$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k \text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)} = \sum_{k \text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x) = 1 + x + x^2 + x^3 + \ldots = \sum_k x^k$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x \frac{d}{dx} \sum_k x^k = x \sum_k k\, x^{k-1} = \sum_k k\, x^k$$

$$x/(1-x)^2$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$e^{-x^2} = \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!}$$

$$\cosh(x) = (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\,(k!)} = \sum_{k\text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!}$$

$$\sinh(x)/x = (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\,(k!)} = \sum_{k\text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!}$$

$$1/(1-x) = 1 + x + x^2 + x^3 + \ldots = \sum_k x^k$$

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x \frac{d}{dx} \sum_k x^k = x \sum_k k\,x^{k-1} = \sum_k k\,x^k$$

$$x/(1-x)^2 = \sum_k k\,x^k.$$

Odd and even functions

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$.

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. For example, $\cos(x)$ is even and $\sin(x)$ is odd.

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. For example, $\cos(x)$ is even and $\sin(x)$ is odd. If

$$f(x) = \sum_k a_k x^k = \sum_{k \text{ even}} a_k x^k + \sum_{k \text{ odd}} a_k x^k$$

then

$$f(-x) = \sum_k a_k (-x)^k = \sum_{k \text{ even}} a_k x^k - \sum_{k \text{ odd}} a_k x^k.$$

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. For example, $\cos(x)$ is even and $\sin(x)$ is odd. If

$$f(x) = \sum_k a_k x^k = \sum_{k \text{ even}} a_k x^k + \sum_{k \text{ odd}} a_k x^k$$

then

$$f(-x) = \sum_k a_k (-x)^k = \sum_{k \text{ even}} a_k x^k - \sum_{k \text{ odd}} a_k x^k.$$

Thus $f(x)$ is even iff the Taylor series involves only even powers of x, and $f(x)$ is odd iff the Taylor series involves only odd powers of x.

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. For example, $\cos(x)$ is even and $\sin(x)$ is odd. If

$$f(x) = \sum_k a_k x^k = \sum_{k \ \text{even}} a_k x^k + \sum_{k \ \text{odd}} a_k x^k$$

then

$$f(-x) = \sum_k a_k (-x)^k = \sum_{k \ \text{even}} a_k x^k - \sum_{k \ \text{odd}} a_k x^k.$$

Thus $f(x)$ is even iff the Taylor series involves only even powers of x, and $f(x)$ is odd iff the Taylor series involves only odd powers of x.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Algebra of series

Algebra of series

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^7)$$

Algebra of series

$$\tan(x) = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + O(x^7)$$

$$\tan(x)^2 = (x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5)^2 + O(x^7)$$

Algebra of series

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^7)$$

$$\begin{aligned}\tan(x)^2 &= (x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5)^2 + O(x^7) \\ &= x^2 + \tfrac{1}{3}x^4 + \tfrac{2}{15}x^6 + \\ &\quad \tfrac{1}{3}x^4 + \tfrac{1}{9}x^6 + \tfrac{2}{45}x^8 \\ &\quad \tfrac{2}{15}x^6 + \tfrac{2}{45}x^8 + \tfrac{4}{225}x^{10} + O(x^7)\end{aligned}$$

Algebra of series

$$\tan(x) = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + O(x^7)$$

$$\begin{aligned}
\tan(x)^2 &= (x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5)^2 + O(x^7) \\
&= x^2 + \tfrac{1}{3}x^4 + \tfrac{2}{15}x^6 + \\
&\quad \tfrac{1}{3}x^4 + \tfrac{1}{9}x^6 + \tfrac{2}{45}x^8 \\
&\quad \tfrac{2}{15}x^6 + \tfrac{2}{45}x^8 + \tfrac{4}{225}x^{10} + O(x^7) \\
&= x^2 + \tfrac{2}{3}x^4 + \tfrac{17}{45}x^6 + O(x^7).
\end{aligned}$$

Expansion about other points

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α.

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k (x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!$$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k (x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!$$

$\ln'(x) = x^{-1}$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k (x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!$$

$\ln'(x) = x^{-1}$ $\qquad$ $\ln''(x) = -x^{-2}$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k (x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!$$

$\ln'(x) = x^{-1} \qquad \ln''(x) = -x^{-2} \qquad \ln'''(x) = 2x^{-3}$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k(x-\alpha)^k, \qquad \text{where} \qquad b_k = f^{(k)}(\alpha)/k!$$

$\ln'(x) = x^{-1}$ $\qquad \ln''(x) = -x^{-2}$ $\qquad \ln'''(x) = 2x^{-3}$ $\qquad \ln^{(4)}(x) = -6x^{-4}$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k (x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!$$

	$\ln'(x) = x^{-1}$	$\ln''(x) = -x^{-2}$	$\ln'''(x) = 2x^{-3}$	$\ln^{(4)}(x) = -6x^{-4}$
$\ln(1) = 0$	$\ln'(1) = 1$	$\ln''(1) = -1$	$\ln'''(1) = 2$	$\ln^{(4)}(1) = -6$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k (x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!$$

$$\begin{array}{lllll}
 & \ln'(x) = x^{-1} & \ln''(x) = -x^{-2} & \ln'''(x) = 2x^{-3} & \ln^{(4)}(x) = -6x^{-4} \\
\ln(1) = 0 & \ln'(1) = 1 & \ln''(1) = -1 & \ln'''(1) = 2 & \ln^{(4)}(1) = -6 \\
b_0 = 0 & b_1 = 1 & b_2 = -1/2 & b_3 = 2/3! = 1/3 & b_4 = -6/4! = -1/4
\end{array}$$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k (x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!$$

	$\ln'(x) = x^{-1}$	$\ln''(x) = -x^{-2}$	$\ln'''(x) = 2x^{-3}$	$\ln^{(4)}(x) = -6x^{-4}$
$\ln(1) = 0$	$\ln'(1) = 1$	$\ln''(1) = -1$	$\ln'''(1) = 2$	$\ln^{(4)}(1) = -6$
$b_0 = 0$	$b_1 = 1$	$b_2 = -1/2$	$b_3 = 2/3! = 1/3$	$b_4 = -6/4! = -1/4$

$$\ln(x) = (x-1) - (x-1)^2/2 + (x-1)^3/3 - (x-1)^4/4 + O((x-1)^5).$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$
$$f'(x) = \frac{1}{\cos(x)^2}$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$

$$f'(x) = \frac{1}{\cos(x)^2}$$

$$f''(x) = -2\cos(x)^{-3}. - \sin(x) = \frac{2\sin(x)}{\cos(x)^3}$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$
\begin{aligned}
f(x) &= \tan(x) \\
f'(x) &= \frac{1}{\cos(x)^2} \\
f''(x) &= -2\cos(x)^{-3}. - \sin(x) = \frac{2\sin(x)}{\cos(x)^3}
\end{aligned}
$$

$f(\frac{\pi}{4}) = 1$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$

$$f'(x) = \frac{1}{\cos(x)^2}$$

$$f''(x) = -2\cos(x)^{-3}. - \sin(x) = \frac{2\sin(x)}{\cos(x)^3}$$

$$f(\tfrac{\pi}{4}) = 1 \qquad f'(\tfrac{\pi}{4}) = \frac{1}{(2^{-1/2})^2} = 2$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$
$$f'(x) = \frac{1}{\cos(x)^2}$$
$$f''(x) = -2\cos(x)^{-3}. - \sin(x) = \frac{2\sin(x)}{\cos(x)^3}$$

$$f(\tfrac{\pi}{4}) = 1 \qquad f'(\tfrac{\pi}{4}) = \frac{1}{(2^{-1/2})^2} = 2 \qquad f''(\tfrac{\pi}{4}) = \frac{2.2^{-1/2}}{(2^{-1/2})^3} = 4$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$
$$f'(x) = \frac{1}{\cos(x)^2}$$
$$f''(x) = -2\cos(x)^{-3}. - \sin(x) = \frac{2\sin(x)}{\cos(x)^3}$$

$$f(\tfrac{\pi}{4}) = 1 \qquad f'(\tfrac{\pi}{4}) = \frac{1}{(2^{-1/2})^2} = 2 \qquad f''(\tfrac{\pi}{4}) = \frac{2.2^{-1/2}}{(2^{-1/2})^3} = 4$$

$$a_0 = 1/0! = 1 \qquad a_1 = 2/1! = 2 \qquad a_2 = 4/2! = 2$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$

$$f'(x) = \frac{1}{\cos(x)^2}$$

$$f''(x) = -2\cos(x)^{-3}. - \sin(x) = \frac{2\sin(x)}{\cos(x)^3}$$

$$f(\tfrac{\pi}{4}) = 1 \qquad f'(\tfrac{\pi}{4}) = \frac{1}{(2^{-1/2})^2} = 2 \qquad f''(\tfrac{\pi}{4}) = \frac{2.2^{-1/2}}{(2^{-1/2})^3} = 4$$

$$a_0 = 1/0! = 1 \qquad a_1 = 2/1! = 2 \qquad a_2 = 4/2! = 2$$

$$\tan(x) = 1 + 2(x - \tfrac{\pi}{4}) + 2(x - \tfrac{\pi}{4})^2 + O((x - \tfrac{\pi}{4})^3).$$

More examples

Consider $y = x/(e^x - 1)$.

More examples

Consider $y = x/(e^x - 1)$.

$$e^x = 1 + x + x^2/2 + x^3/6 + O(x^4)$$

More examples

Consider $y = x/(e^x - 1)$.

$$\begin{aligned} e^x &= 1 + x + x^2/2 + x^3/6 + O(x^4) \\ e^x - 1 &= x + x^2/2 + x^3/6 + O(x^4) \end{aligned}$$

More examples

Consider $y = x/(e^x - 1)$.

$$\begin{aligned} e^x &= 1 + x + x^2/2 + x^3/6 + O(x^4) \\ e^x - 1 &= x + x^2/2 + x^3/6 + O(x^4) \\ \frac{1}{y} = \frac{e^x - 1}{x} &= 1 + x/2 + x^2/6 + O(x^3) \end{aligned}$$

More examples

Consider $y = x/(e^x - 1)$.

$$e^x = 1 + x + x^2/2 + x^3/6 + O(x^4)$$

$$e^x - 1 = x + x^2/2 + x^3/6 + O(x^4)$$

$$\frac{1}{y} = \frac{e^x - 1}{x} = 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6$$

More examples

Consider $y = x/(e^x - 1)$.

$$\begin{aligned} e^x &= 1 + x + x^2/2 + x^3/6 + O(x^4) \\ e^x - 1 &= x + x^2/2 + x^3/6 + O(x^4) \\ \frac{1}{y} = \frac{e^x - 1}{x} &= 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6 \\ y = \frac{1}{1+u} &= 1 - u + u^2 + O(u^3) = 1 - u + u^2 + O(x^3) \end{aligned}$$

More examples

Consider $y = x/(e^x - 1)$.

$$e^x = 1 + x + x^2/2 + x^3/6 + O(x^4)$$

$$e^x - 1 = x + x^2/2 + x^3/6 + O(x^4)$$

$$\frac{1}{y} = \frac{e^x - 1}{x} = 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6$$

$$y = \frac{1}{1+u} = 1 - u + u^2 + O(u^3) = 1 - u + u^2 + O(x^3)$$

$$u^2 = x^2/4 + x^3/6 + x^4/36$$

More examples

Consider $y = x/(e^x - 1)$.

$$e^x = 1 + x + x^2/2 + x^3/6 + O(x^4)$$

$$e^x - 1 = x + x^2/2 + x^3/6 + O(x^4)$$

$$\frac{1}{y} = \frac{e^x - 1}{x} = 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6$$

$$y = \frac{1}{1+u} = 1 - u + u^2 + O(u^3) = 1 - u + u^2 + O(x^3)$$

$$u^2 = x^2/4 + x^3/6 + x^4/36 = x^2/4 + O(x^3)$$

More examples

Consider $y = x/(e^x - 1)$.

$$\begin{aligned} e^x &= 1 + x + x^2/2 + x^3/6 + O(x^4) \\ e^x - 1 &= x + x^2/2 + x^3/6 + O(x^4) \\ \frac{1}{y} = \frac{e^x - 1}{x} &= 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6 \\ y = \frac{1}{1+u} &= 1 - u + u^2 + O(u^3) = 1 - u + u^2 + O(x^3) \\ u^2 &= x^2/4 + x^3/6 + x^4/36 = x^2/4 + O(x^3) \\ \frac{x}{e^x - 1} &= 1 - u + u^2 + O(x^3) \end{aligned}$$

More examples

Consider $y = x/(e^x - 1)$.

$$\begin{aligned}
e^x &= 1 + x + x^2/2 + x^3/6 + O(x^4) \\
e^x - 1 &= x + x^2/2 + x^3/6 + O(x^4) \\
\frac{1}{y} = \frac{e^x - 1}{x} &= 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6 \\
y = \frac{1}{1+u} &= 1 - u + u^2 + O(u^3) = 1 - u + u^2 + O(x^3) \\
u^2 &= x^2/4 + x^3/6 + x^4/36 = x^2/4 + O(x^3) \\
\frac{x}{e^x - 1} &= 1 - u + u^2 + O(x^3) \\
&= 1 - x/2 - x^2/6 + x^2/4 + O(x^3)
\end{aligned}$$

More examples

Consider $y = x/(e^x - 1)$.

$$\begin{aligned}
e^x &= 1 + x + x^2/2 + x^3/6 + O(x^4) \\
e^x - 1 &= x + x^2/2 + x^3/6 + O(x^4) \\
\frac{1}{y} = \frac{e^x - 1}{x} &= 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6 \\
y = \frac{1}{1+u} &= 1 - u + u^2 + O(u^3) = 1 - u + u^2 + O(x^3) \\
u^2 &= x^2/4 + x^3/6 + x^4/36 = x^2/4 + O(x^3) \\
\frac{x}{e^x - 1} &= 1 - u + u^2 + O(x^3) \\
&= 1 - x/2 - x^2/6 + x^2/4 + O(x^3) = 1 - x/2 + x^2/12 + O(x^3)
\end{aligned}$$