Special functions

December 10, 2009

Introduction

The *primary special functions* are

$$\exp, \ln, \sin, \cos, \tan, \arcsin, \arccos, \arctan.$$

Things you should know:

- The detailed shape of the graphs
- Domains, ranges and inverses
- Properties such as $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- Derivatives and integrals (covered in later lectures).

The *secondary special functions* are

$$\text{sec}, \text{csc}, \cot, \sinh, \cosh, \tanh,$$
$$\text{sech}, \text{csch}, \coth, \text{arcsinh}, \text{arccosh}, \text{arctanh}.$$

- You should know how these are defined in terms of the primary functions (for example, $\sinh(x) = (\exp(x) - \exp(-x))/2$, and $\sec(x) = 1/\cos(x)$)
- You should either remember the properties of the secondary functions, or be able to derive them from the properties of the primary functions

The exponential function

- $\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
  Warning: infinite sums are subtle.
- $e = \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \simeq 2.71828.$
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$$\begin{aligned} \exp(x+y) &= \exp(x)\exp(y) & \exp(x-y) &= \exp(x)/\exp(y) \\ \exp(0) &= 1 & \exp(-x) &= 1/\exp(x) \\ \exp(nx) &= \exp(x)^n & \exp(x) &= e^x \end{aligned}$$

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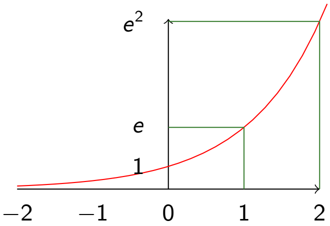
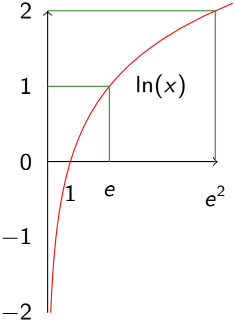


The logarithm

- The natural log function $\ln(y)$ is the inverse of the exponential.
- $\ln(y)$ is defined only when $y > 0$ (unless we use complex numbers).
- We have $\ln(\exp(x)) = \ln(e^x) = x$ for all x, and $\exp(\ln(y)) = e^{\ln(y)} = y$ when $y > 0$ (*NOT* $\ln(x) = 1/\exp(x)$).
- $$\begin{aligned} \ln(xy) &= \ln(x) + \ln(y) & \qquad \ln(x/y) &= \ln(x) - \ln(y) \\ \ln(1) &= 0 & \ln(1/y) &= -\ln(y) \\ \ln(y^n) &= n\ln(y) & \ln(e) &= 1. \end{aligned}$$
- 

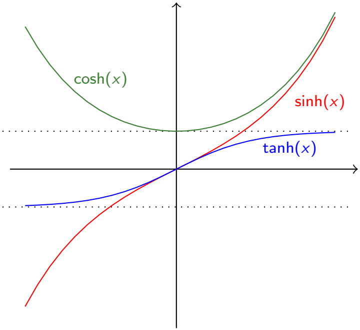
Logs to other bases

- $\log_a(y)$ is the number t such that $y = a^t$ (defined for $a, y > 0$).
- $$\log_{10}(1000) = \log_{10}(10^3) = 3$$
  $$\log_2(1024) = \log_2(2^{10}) = 10$$
  $$\log_{1024}(2) = \log_{1024}(1024^{1/10}) = 1/10$$
  $$\log_3(1/9) = \log_3(3^{-2}) = -2$$
- $\log_a(y) = \ln(y)/\ln(a)$
- Check: $a^{\ln(y)/\ln(a)} = (e^{\ln(a)})^{\ln(y)/\ln(a)} = e^{\ln(y)} = y$.
- $\log_{10}(y) =$ the number t such that $10^t = y$
  $\simeq$ the number of digits in y left of the decimal point.
- This is mostly of historical importance.
- $\log_2(y) =$ the number t such that $2^t = y$
  $\simeq$ the number of bits in y.
- This is of some use in computer science and information theory.
- $\log_e(y) =$ (the number t such that $e^t = y$) $= \ln(y) = \log(y)$.

Hyperbolic functions

- The hyperbolic functions are defined as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

  Use `convert(...,exp)` in Maple to rewrite in terms of exponentials.
- Properties are easily deduced from those of exp.
- These are related to trig functions using complex numbers, eg $\sin(x) = \sinh(ix)/i$, where $i = \sqrt{-1}$.
- 

Hyperbolic identities

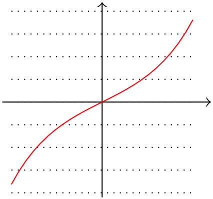
- $$\cosh(x)^2 - \sinh(x)^2 = 1$$
  $$\operatorname{sech}(x)^2 + \tanh(x)^2 = 1$$
  $$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$
  $$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$
- To check these, put $u = e^x$, so $\sinh(x) = \frac{u-u^{-1}}{2}$ and $\cosh(x) = \frac{u+u^{-1}}{2}$.
- $$\begin{aligned}\cosh(x)^2 - \sinh(x)^2 &= \frac{(u+u^{-1})^2}{4} - \frac{(u-u^{-1})^2}{4} \\ &= \frac{(u^2 + 2 + u^{-2}) - (u^2 - 2 + u^{-2})}{4} \\ &= (2-(-2))/4 = 1.\end{aligned}$$
- Now put $v = e^y$, so $uv = e^{x+y}$.
- $\sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \frac{(u-u^{-1})}{2}\frac{(v+v^{-1})}{2} + \frac{(u+u^{-1})}{2}\frac{(v-v^{-1})}{2}$

$$= \frac{(uv + uv^{-1} - u^{-1}v - u^{-1}v^{-1} + uv - uv^{-1} + u^{-1}v - u^{-1}v^{-1})}{4}$$

$$= \frac{uv - (uv)^{-1}}{2} = \frac{e^{x+y} - e^{-x-y}}{2} = \sinh(x+y)$$

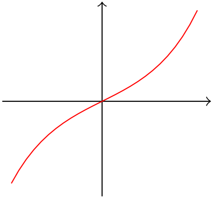
Inverse hyperbolic functions

- The graph of $y = \sinh(x)$ crosses each horizontal line precisely once, which means that there is an inverse function $x = \sinh^{-1}(y) = \operatorname{arcsinh}(y)$, defined for all $y \in \mathbb{R}$.

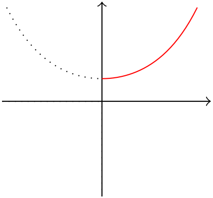


- This can be written in terms of ln: $\operatorname{arcsinh}(y) = \ln(y + \sqrt{1+y^2})$.
- **Check:** Suppose $y = \sinh(x)$; we must show that $x = \ln(y + \sqrt{1+y^2})$.
  - We have $1 + y^2 = 1 + \sinh(x)^2 = \cosh(x)^2$ (and $\cosh(x), 1 + y^2 > 0$), so $\sqrt{1+y^2} = \cosh(x)$.
  - Thus $y + \sqrt{1+y^2} = \sinh(x) + \cosh(x) = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = e^x$
  - so $\ln(y + \sqrt{1+y^2}) = \ln(e^x) = x$ as required.
- Similarly, $\operatorname{arccosh}(y) = \ln(y + \sqrt{y^2 - 1})$, defined for $y \geq 1$
- and $\operatorname{arctanh}(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$, defined when $-1 < y < 1$.

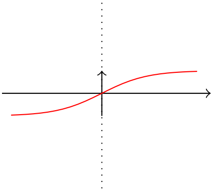
Graphs



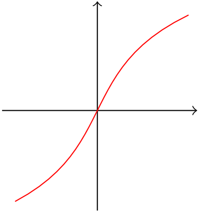
$\sinh(x)$



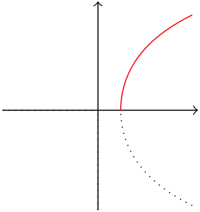
$\cosh(x)$



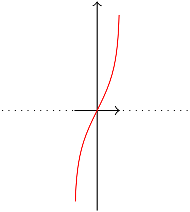
$\tanh(x)$



$\operatorname{arcsinh}(x)$



$\operatorname{arccosh}(x)$



$\operatorname{arctanh}(x)$