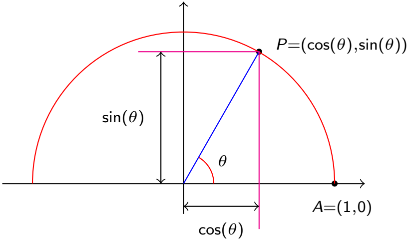
Special functions

December 10, 2009

Trigonometric functions

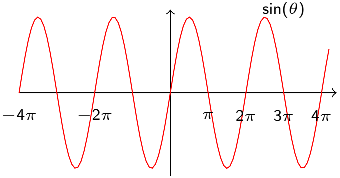
- Let P be one unit away from the origin, at an angle of θ measured anticlockwise from the point $A = (1, 0)$.

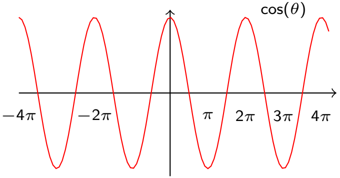


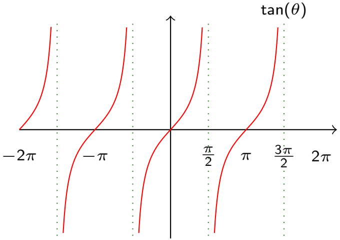
- (We measure θ in radians, so the length of the arc AP is θ.)
- The numbers $\cos(\theta)$ and $\sin(\theta)$ are *defined* to be the x and y coordinates of P.
- We also put

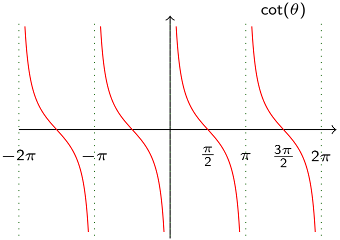
$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$

Graphs









$$
\begin{aligned}
\sin(\pi/2 + x) &= \cos(x) & \cos(\pi/2 + x) &= -\sin(x)\\
\sin(\pi + x) &= -\sin(x) & \cos(\pi + x) &= -\cos(x)\\
\sin(2\pi + x) &= \sin(x) & \cos(2\pi + x) &= \cos(x)\\
\sin(-x) &= -\sin(x) & \cos(-x) &= \cos(x).
\end{aligned}
$$

Preview of complex numbers

- Complex numbers are expressions like $z = 3 + 4i$, where i satisfies $i^2 = -1$.
- You can add and subtract complex numbers in an obvious way, for example $(3 + 4i) + (7 - 3i) = 10 + i$.
- To multiply: expand out and use $i^2 = -1$. For example: $(1 + 2i)(3 + 4i) = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 = -5 + 10i$.
- Note that the powers of i repeat with period 4:

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad i^6 = -1 \quad i^7 = -i \quad i^8 = 1.$$

- By expanding and using this we find powers of any complex number.

$$(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i + (-1) = 2i$$
$$(1 + i)^8 = ((1 + i)^2)^4 = 2^4 i^4 = 2^4 = 16$$

- Note that

$$\begin{aligned}
\exp(ix) =& 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{6} + \frac{(ix)^4}{24} + \frac{(ix)^5}{120} + \cdots \\
&= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + i\frac{x^5}{120} + \cdots \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) i \\
&= \cos(x) + \sin(x) i.
\end{aligned}$$

De Moivre's theorem

$$e^{i\theta} = \exp(i\theta) = \cos(\theta) + \sin(\theta) i$$

$$\begin{aligned}
e^{-i\theta} &= \exp(-i\theta) = \cos(\theta) - \sin(\theta) i \\
\sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i} &&= \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sinh(i\theta)/i \\
\cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} &&= \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh(i\theta) \\
\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sinh(i\theta)/i}{\cosh(i\theta)} = \tanh(i\theta)/i.
\end{aligned}$$

$$\begin{aligned}
\cos(a)^2 + \sin(a)^2 &= 1 \\
\sec(a)^2 &= 1 + \tan(a)^2 \\
\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\
\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
\sin(2a) &= 2\sin(a)\cos(a) \\
\cos(2a) &= 2\cos(a)^2 - 1 = 1 - 2\sin(a)^2.
\end{aligned}$$

Examples

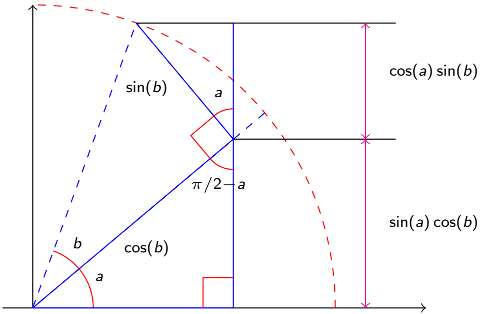
$$\cos(a)^2 + \sin(a)^2 = \left(\frac{e^{ia} + e^{-ia}}{2}\right)^2 + \left(\frac{e^{ia} - e^{-ia}}{2i}\right)^2$$
$$= (e^{2ia} + 2e^{ia-ia} + e^{-2ia})/4 + (e^{2ia} - 2e^{ia-ia} + e^{-2ia})/(-4) = ({\color{red}e^{2ia}} +$$
$$= 2/4 - 2/(-4) = 1$$

$$\cos(a)^2 - \sin(a)^2 = \left(\frac{e^{ia} + e^{-ia}}{2}\right)^2 - \left(\frac{e^{ia} - e^{-ia}}{2i}\right)^2$$
$$= (e^{2ia} + 2 + e^{-2ia})/4 + (e^{2ia} - 2 + e^{-2ia})/4$$
$$= 2(e^{2ia} + e^{-2ia})/4 = (e^{2ia} + e^{-2ia})/2 = \cos(2a)$$

$$2\sin(a)\cos(a) = 2\left(\frac{e^{ia} - e^{-ia}}{2i}\right)\left(\frac{e^{ia} + e^{-ia}}{2}\right)$$
$$= \frac{2}{4i}\left(e^{2ia} + e^{0} - e^{0} - e^{-2ia}\right) = (e^{2ia} - e^{-2ia})/(2i) = \sin(2a)$$

The addition formula

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$



$$\begin{aligned}\sin(a)\cos(b) + \cos(a)\sin(b) &= \frac{e^{ia} - e^{-ia}}{2i}\,\frac{e^{ib} + e^{-ib}}{2} + \frac{e^{ia} + e^{-ia}}{2}\,\frac{e^{ib} - e^{-ib}}{2i} \\ &= \frac{e^{i(a+b)} - e^{-i(a+b)}}{2i} = \sin(a+b)\end{aligned}$$

Finite Fourier series

- A *finite Fourier series* is a sum of constant multiples of functions of the form $\sin(nx)$ or $\cos(mx)$ (with $n, m \in \mathbb{Z}$). Note that the constant function $f(x) = a = a\cos(0x)$ is included.
- The phrase *trigonometric polynomial* means the same thing.
- Many functions can be rewritten as finite Fourier series:

$$\begin{aligned}
\sin(x)^2 &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) \\
\sin(x)^3 &= \tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x) \\
\sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\
\sin(x)^4 + \cos(x)^4 &= \tfrac{3}{4} + \tfrac{1}{4}\cos(4x) \\
\sin(nx)\sin(mx) &= \tfrac{1}{2}\cos((n-m)x) - \tfrac{1}{2}\cos((n+m)x).
\end{aligned}$$

- **Method:** Rewrite using $\cos(n\theta) = (e^{in\theta} + e^{-in\theta})/2$ and $\sin(n\theta) = (e^{in\theta} - e^{-in\theta})/2i$, expand out, then rewrite using $e^{im\theta} = \cos(m\theta) + \sin(m\theta)i$.
- Once a function has been rewritten in this form, it is very easy to differentiate it or integrate it.

Examples

Problem: write $\sin(x)^4 + \cos(x)^4$ as a Fourier series.

Put $u = e^{ix}$, so $\sin(x) = (u - u^{-1})/(2i)$ and $\cos(x) = (u + u^{-1})/2$. Note that $i^2 = -1$ so $i^4 = (-1)^2 = 1$ so $(2i)^4 = 2^4 = 16$. Note also that

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

(use the binomial formula, or expand it out.) Thus

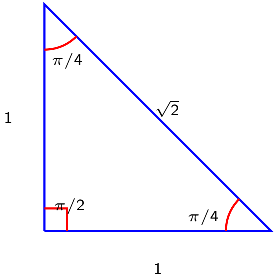
$$\begin{aligned}
\sin(x)^4 + \cos(x)^4 &= (u - u^{-1})^4/16 + (u + u^{-1})^4/16 \\
&= (u^4 - 4u^2 + 6 - 4u^{-2} + u^{-4})/16 + \\
&\quad (u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4})/16 \\
&= 12/16 + 2(u^4 + u^{-4})/16 = 3/4 + ((u^4 + u^{-4})/2)/4 \\
&= (3 + \cos(4x))/4
\end{aligned}$$

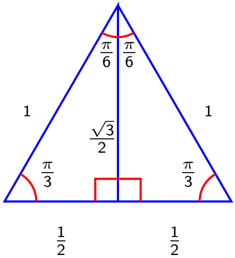
Special values

You should know the following values of $\sin(\theta)$ and $\cos(\theta)$:

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\pi/2$	1	0	∞
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$

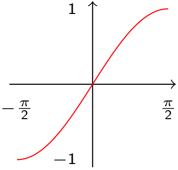
Proved by considering these triangles:



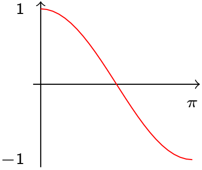


You should also be able to deduce things like $\cos(5\pi/6) = -\sqrt{3}/2$.

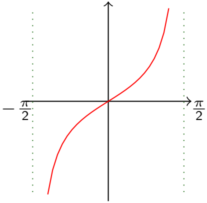
Inverse trigonometric functions



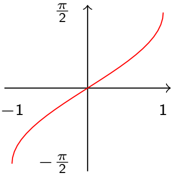
$\sin\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1,1]$



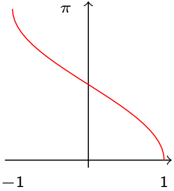
$\cos\colon [0,\pi] \longrightarrow [-1,1]$



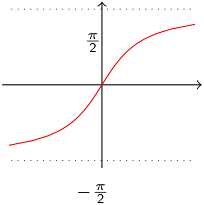
$\tan\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R}$



$\arcsin\colon [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\arccos\colon [-1,1] \longrightarrow [0,\pi]$



$\arctan\colon \mathbb{R} \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$