Differentiation

Introduction

Things you should know:

- The meaning of differentiation (slopes of graphs, time-dependent and space-dependent variables, etc)
- Some derivatives from first principles: x^2, $1/x$, e^x.
- Rules for finding derivatives:
  - The product rule $((uv)' = u'v + uv')$
  - The quotient rule $((u/v)' = (u'v - uv')/v^2)$
  - The chain rule $(\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx})$
  - The power rule $((u^n)' = nu^{n-1}u')$
  - The logarithmic rule $(\log(u)' = u'/u)$
  - The inverse function rule $(\frac{dx}{dy} = 1/\frac{dy}{dx})$
- Derivatives of various classes of functions (eg the derivative of a rational function is another rational function.)

You must learn to find derivatives quickly and accurately.

Meaning

- Consider related variables x and y; so whenever x changes, so does y.
- Examples:
  - $p =$ price of chocolate ; $d =$ demand for chocolate .
  - $t =$ time ; $d =$ atmospheric CO_2 concentration .
  - $r =$ distance from sun ; $g =$ strength of solar gravity .
- If x changes to $x + \delta x$, then y changes to $y + \delta y$.

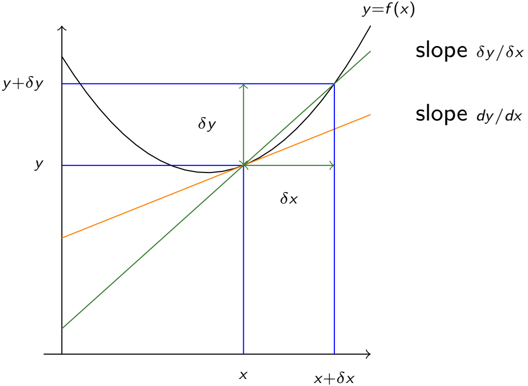
$$\frac{dy}{dx} = \lim_{\delta x \longrightarrow 0} \frac{\delta y}{\delta x} = \text{ derivative of } y \text{ with respect to } x.$$

- If $y = f(x)$, then $\delta y = f(x + \delta x) - f(x)$, so

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \longrightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{h \longrightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

- We sometimes write y' for dy/dx (care needed).

Slopes



Consider variables x and y related by $y = f(x)$.dy/dx is the slope of the tangent line to the graph.If x changes by a small amount δx, then y will change by a small amount δy.The ratio $\delta y/\delta x$ is the slope of a chord cutting across the graph.The slope of the chord changes slightly as δx decreases.As δx approaches zero, the chord approaches the tangent, and $\delta y/\delta x$ approaches dy/dx.

The function $f(x) = x^2$

- Consider the function $f(x) = x^2$.
- Then $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$, so

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\
&= \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \frac{2xh + h^2}{h} \\
&= 2x + h
\end{aligned}$$

- Thus

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \longrightarrow 0} (2x + h) = 2x.$$

- Similarly:

$$\boxed{\tfrac{d}{dx}(x^n) = nx^{n-1} \text{ for all } n.}$$

The function $f(x) = 1/x$

- Consider the function $f(x) = 1/x$.
- $$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x(x+h)}$$

  so
  $$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x(x+h)}$$

  so
  $$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \longrightarrow 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

The exponential function

- Consider the function $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.
- $$f(x+h) - f(x) = e^{x+h} - e^x = e^x(e^h - 1) = e^x\left(h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots\right)$$

  so
  $$\frac{f(x+h) - f(x)}{h} = e^x\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right)$$
  so
  $$\begin{aligned} f'(x) &= \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \longrightarrow 0} e^x\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right) \\ &= e^x(1 + 0 + 0 + \cdots) \\ &= e^x. \end{aligned}$$

- Conclusion: $\exp'(x) = \exp(x)$.

Special functions

$$
\begin{aligned}
\exp'(x) &= \exp(x) \\
\sinh'(x) &= \cosh(x) \\
\cosh'(x) &= \sinh(x) \\
\tanh'(x) &= \operatorname{sech}(x)^2 = 1 - \tanh(x)^2 \\
\sin'(x) &= \cos(x) \\
\cos'(x) &= -\sin(x) \\
\tan'(x) &= \sec(x)^2 = 1 + \tan(x)^2
\end{aligned}
$$

$$
\begin{aligned}
\log'(x) &= 1/x \\
\operatorname{arcsinh}'(x) &= (1 + x^2)^{-1/2} \\
\operatorname{arccosh}'(x) &= (x^2 - 1)^{-1/2} \\
\operatorname{arctanh}'(x) &= (1 - x^2)^{-1} \\
\arcsin'(x) &= (1 - x^2)^{-1/2} \\
\arccos'(x) &= -(1 - x^2)^{-1/2} \\
\arctan'(x) &= (1 + x^2)^{-1}
\end{aligned}
$$

- We showed earlier that $\exp'(x) = \exp(x)$
- We deduce $\sinh'(x)$ using the identity $\sinh(x) = (e^x - e^{-x})/2$. Similarly for cosh and tanh.
- Using $\cos(x) = \cosh(ix)$ etc, we find $\sin'(x)$, $\cos'(x)$ and $\tan'(x)$.
- Using $\exp'(x) = \exp(x)$ and the inverse function rule, we find that $\log'(x) = 1/x$
- The inverse function rule also gives the remaining derivatives.

The product rule

- Consider variables u and v depending on x, and put $w = uv$. Then

$$\boxed{w' = (uv)' = u'v + uv'} \qquad \boxed{\tfrac{dw}{dx} = \tfrac{d}{dx}(uv) = \tfrac{du}{dx}v + u\tfrac{dv}{dx}.}$$

- If x changes to $x + \delta x$, then u changes to $u + \delta u$ & v changes to $v + \delta v$ so w changes to

$$\begin{aligned}
w + \delta w &= (u + \delta u)(v + \delta v) = uv + (\delta u)\,v + u\,(\delta v) + (\delta u)(\delta v) \\
\delta w &= (\delta u)\,v + u\,(\delta v) + (\delta u)(\delta v) \\
\frac{\delta w}{\delta x} &= \frac{\delta u}{\delta x}v + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x \\
&\simeq \frac{du}{dx}v + u\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx}\delta x \simeq \frac{du}{dx}v + u\frac{dv}{dx}
\end{aligned}$$

(The approximations become exact in the limit as $\delta x \to 0$.)

Examples of the product rule

$$
\begin{aligned}
(uv)' &= u'v + uv' \\
\frac{d}{dx}(\sin(x)\cos(x)) &= \sin'(x)\cos(x) + \sin(x)\cos'(x) \\
&= \cos(x)\cos(x) + \sin(x)(-\sin(x)) \\
&= \cos(x)^2 - \sin(x)^2 \\
\frac{d}{dx}(x^3\log(x)) &= 3x^2\log(x) + x^3\log'(x) \\
&= 3x^2\log(x) + x^3(x^{-1}) \\
&= (3\log(x) + 1)x^2 \\
\frac{d}{dx}(e^{ax}\sin(bx)) &= a\,e^{ax}\sin(bx) + e^{ax}\,b\cos(bx) \\
&= e^{ax}(a\sin(bx) + b\cos(bx))
\end{aligned}
$$

The quotient rule

- Consider variables u and v depending on x, and put $w = u/v$. Then

$$\boxed{w' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

- Indeed: $u = vw$, so $u' = v'w + vw'$ (product rule) , so

$$w' = \frac{u' - v'w}{v} = \frac{u'}{v} - \frac{v'.(u/v)}{v} = \frac{u'}{v} - \frac{uv'}{v^2} = \frac{u'v - uv'}{v^2}.$$

Examples of the quotient rule

$$\frac{d}{dx}\left(\frac{x}{\log(x)}\right) = \frac{1.\log(x) - xx^{-1}}{\log(x)^2} = \frac{\log(x)-1}{\log(x)^2} = \log(x)^{-1} - \log(x)^{-2}$$

(Aside: $x/\log(x) \simeq$ (number of primes $\leq x$))

$$\frac{d}{dx}\left(\frac{x}{1-x^2}\right) = \frac{1.(1-x^2) - x.(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

Now consider $\tan'(x)$, remembering that $\tan(x) = \sin(x)/\cos(x)$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) &= \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos(x)^2} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos(x)^2} \\ &= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2} = \sec(x)^2\end{aligned}$$