

Differentiation

Things you should know:

- ▶ The meaning of differentiation (slopes of graphs, time-dependent and space-dependent variables, etc)
- ▶ Some derivatives from first principles: x^2 , $1/x$, e^x .
- ▶ Rules for finding derivatives:
 - ▶ The product rule $((uv)') = u'v + uv'$
 - ▶ The quotient rule $((u/v)') = (u'v - uv')/v^2$
 - ▶ The chain rule $(\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx})$
 - ▶ The power rule $((u^n)') = nu^{n-1}u'$
 - ▶ The logarithmic rule $(\log(u)') = u'/u$
 - ▶ The inverse function rule $(\frac{dx}{dy} = 1/\frac{dy}{dx})$
- ▶ Derivatives of various classes of functions (eg the derivative of a rational function is another rational function.)

You must learn to find derivatives quickly and accurately.

- ▶ Consider related variables x and y ; so whenever x changes, so does y .
- ▶ Examples:
 - ▶ p = price of chocolate ; d = demand for chocolate .
 - ▶ t = time ; d = atmospheric CO_2 concentration .
 - ▶ r = distance from sun ; g = strength of solar gravity .
- ▶ If x changes to $x + \delta x$, then y changes to $y + \delta y$.

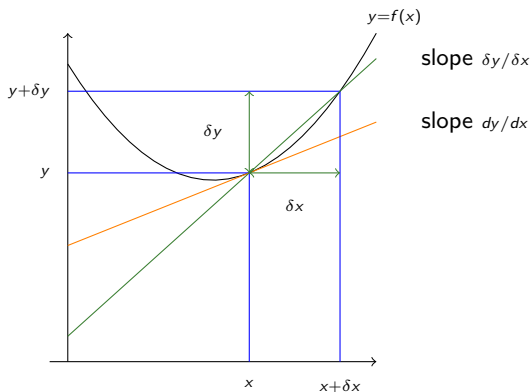
$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \text{derivative of } y \text{ with respect to } x.$$

- ▶ If $y = f(x)$, then $\delta y = f(x + \delta x) - f(x)$, so

$$f'(x) = \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

- ▶ We sometimes write y' for dy/dx (**care needed**).

Slopes



Consider variables x and y related by $y = f(x)$. dy/dx is the slope of the tangent line to the graph. If x changes by a small amount δx , then y will change by a small amount δy . The ratio $\delta y / \delta x$ is the slope of a chord cutting across the graph. The slope of the chord changes slightly as δx decreases. As δx approaches zero, the chord approaches the tangent, and $\delta y / \delta x$ approaches dy/dx .

The function $f(x) = x^2$

- ▶ Consider the function $f(x) = x^2$.
- ▶ Then $f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$, so

$$\begin{aligned}\frac{f(x + h) - f(x)}{h} &= \frac{(x + h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h\end{aligned}$$

- ▶ Thus

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

- ▶ Similarly:

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1} \text{ for all } n.}$$

The function $f(x) = 1/x$

- ▶ Consider the function $f(x) = 1/x$.

- ▶
$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{-h}{x(x+h)}$$

so

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x(x+h)}$$

so

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

The exponential function

▶ Consider the function $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$.

▶ $f(x+h) - f(x) = e^{x+h} - e^x = e^x(e^h - 1) = e^x \left(h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots \right)$

so

$$\frac{f(x+h) - f(x)}{h} = e^x \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \dots \right)$$

so

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} e^x \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \dots \right) \\ &= e^x (1 + 0 + 0 + \dots) \\ &= e^x. \end{aligned}$$

▶ Conclusion: $\exp'(x) = \exp(x)$.

$\exp'(x)$	$= \exp(x)$	$\log'(x)$	$= 1/x$
$\sinh'(x)$	$= \cosh(x)$	$\operatorname{arcsinh}'(x)$	$= (1+x^2)^{-1/2}$
$\cosh'(x)$	$= \sinh(x)$	$\operatorname{arccosh}'(x)$	$= (x^2-1)^{-1/2}$
$\tanh'(x)$	$= \operatorname{sech}(x)^2 = 1 - \tanh(x)^2$	$\operatorname{arctanh}'(x)$	$= (1-x^2)^{-1}$
$\sin'(x)$	$= \cos(x)$	$\arcsin'(x)$	$= (1-x^2)^{-1/2}$
$\cos'(x)$	$= -\sin(x)$	$\operatorname{arccos}'(x)$	$= -(1-x^2)^{-1/2}$
$\tan'(x)$	$= \sec(x)^2 = 1 + \tan(x)^2$	$\operatorname{arctan}'(x)$	$= (1+x^2)^{-1}$

- ▶ We showed earlier that $\exp'(x) = \exp(x)$
- ▶ We deduce $\sinh'(x)$ using the identity $\sinh(x) = (e^x - e^{-x})/2$. Similarly for \cosh and \tanh .
- ▶ Using $\cos(x) = \cosh(ix)$ etc, we find $\sin'(x)$, $\cos'(x)$ and $\tan'(x)$.
- ▶ Using $\exp'(x) = \exp(x)$ and the inverse function rule, we find that $\log'(x) = 1/x$
- ▶ The inverse function rule also gives the remaining derivatives.

The product rule

- ▶ Consider variables u and v depending on x , and put $w = uv$. Then

$$w' = (uv)' = u'v + uv'$$

$$\frac{dw}{dx} = \frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

- ▶ If x changes to $x + \delta x$, then u changes to $u + \delta u$ & v changes to $v + \delta v$ so w changes to

$$w + \delta w = (u + \delta u)(v + \delta v) = uv + (\delta u)v + u(\delta v) + (\delta u)(\delta v)$$

$$\delta w = (\delta u)v + u(\delta v) + (\delta u)(\delta v)$$

$$\frac{\delta w}{\delta x} = \frac{\delta u}{\delta x}v + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x$$

$$\simeq \frac{du}{dx}v + u\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx}\delta x \simeq \frac{du}{dx}v + u\frac{dv}{dx}$$

(The approximations become exact in the limit as $\delta x \rightarrow 0$.)

Examples of the product rule

$$(uv)' = u'v + uv'$$

$$\begin{aligned}\frac{d}{dx}(\sin(x)\cos(x)) &= \sin'(x)\cos(x) + \sin(x)\cos'(x) \\ &= \cos(x)\cos(x) + \sin(x)(-\sin(x)) \\ &= \cos(x)^2 - \sin(x)^2\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(x^3\log(x)) &= 3x^2\log(x) + x^3\log'(x) \\ &= 3x^2\log(x) + x^3(x^{-1}) \\ &= (3\log(x) + 1)x^2\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(e^{ax}\sin(bx)) &= ae^{ax}\sin(bx) + e^{ax}b\cos(bx) \\ &= e^{ax}(a\sin(bx) + b\cos(bx))\end{aligned}$$

The quotient rule

- ▶ Consider variables u and v depending on x , and put $w = u/v$. Then

$$w' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

- ▶ Indeed: $u = vw$, so $u' = v'w + vw'$ (product rule), so

$$w' = \frac{u' - v'w}{v} = \frac{u'}{v} - \frac{v' \cdot (u/v)}{v} = \frac{u'}{v} - \frac{uv'}{v^2} = \frac{u'v - uv'}{v^2}.$$

Examples of the quotient rule

$$\frac{d}{dx} \left(\frac{x}{\log(x)} \right) = \frac{1 \cdot \log(x) - x x^{-1}}{\log(x)^2} = \frac{\log(x) - 1}{\log(x)^2} = \log(x)^{-1} - \log(x)^{-2}$$

(Aside: $x/\log(x) \simeq$ (number of primes $\leq x$))

$$\frac{d}{dx} \left(\frac{x}{1-x^2} \right) = \frac{1 \cdot (1-x^2) - x \cdot (-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

Now consider $\tan'(x)$, remembering that $\tan(x) = \sin(x)/\cos(x)$.

$$\begin{aligned} \frac{d}{dx} \left(\frac{\sin(x)}{\cos(x)} \right) &= \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\cos(x)^2} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos(x)^2} \\ &= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2} = \sec(x)^2 \end{aligned}$$