Differentiation

The chain rule

- Suppose that y depends on u, and u depends on x. Then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

- If x changes to $x + \delta x$, then u changes to $u + \delta u$ and y changes to $y + \delta y$. Clearly

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u}\frac{\delta u}{\delta x}.$$

  In the limit, δx, δu and δy all approach zero, and we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

- Alternative notation: suppose that $f(x) = g(h(x))$. Then

$$\boxed{f'(x) = g'(h(x))h'(x)}$$

Examples of the chain rule

- Consider $y = \cos(x^2)$. This is $y = \cos(u)$, where $u = x^2$.

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = -\sin(u) = -\sin(x^2)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\sin(x^2).2x = -2x\sin(x^2).$$

- Consider $f(x) = \exp(\sin(x))$.

$$f'(x) = \exp'(\sin(x)).\sin'(x) = \exp(\sin(x))\cos(x).$$

- Consider $y = a\sin(bx + c)$. Put $u = bx + c$, so $y = a\sin(u)$.
  Then $\frac{du}{dx} = b$ and $\frac{dy}{du} = a\cos(u)$ so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = a\cos(u).b = ab\cos(u) = ab\cos(bx + c).$$

The power rule

- If u depends on x and n does not, then

$$\boxed{\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}}$$

- Reason: If $y = u^n$ then $\frac{dy}{du} = nu^{n-1}$ so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}$
- Consider $y = \sqrt{1+x^2}$. This is $y = u^{1/2}$, where $u = 1 + x^2$. Then

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{1+x^2}} \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}}2x = \frac{x}{\sqrt{1+x^2}}.$$

- $\frac{d}{dx}\left(\sin(x)^5\right) = 5\sin(x)^4\cos(x)$
- $\frac{d}{dx}\left(\log(x)^3\right) = 3\log(x)^2x^{-1} = 3\log(x)^2/x$

The logarithmic rule

- $$\boxed{\frac{d}{dx}\log(u) = \frac{1}{u}\,\frac{du}{dx}} \qquad \boxed{\frac{du}{dx} = u\,\frac{d}{dx}\log(u)}$$
- $$\frac{d}{dx}\log(\cos(x)) = \frac{1}{\cos(x)}\cos'(x) = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$
- $$\frac{d}{dx}\log(1+x^2) = \frac{\frac{d}{dx}(1+x^2)}{1+x^2} = \frac{2x}{1+x^2}$$
- Consider $y = x^x$, so $\log(y) = x\log(x)$. Then

$$\begin{aligned}\frac{d}{dx}\log(y) &= \frac{d}{dx}(x\log(x))\\ &= 1.\log(x) + x.x^{-1} = \log(x)+1\\ \frac{dy}{dx} &= y\frac{d}{dx}\log(y)\\ &= x^x(\log(x)+1).\end{aligned}$$

The inverse function rule

- If x and y are interdependent variables, then

$$\boxed{\frac{dx}{dy} = 1\Big/\frac{dy}{dx}}$$

- (Take limits in the obvious relation $\frac{\delta x}{\delta y} = 1/\frac{\delta y}{\delta x}$.)
- Consider $y = \log(x)$, so $x = e^y$.

$$\frac{dx}{dy} = e^y = x \qquad\qquad \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{x}$$

- Alternative notation: if $y = g(x)$ then $x = f(y)$, where $f = g^{-1}$ and $g = f^{-1}$. Then

$$\boxed{g'(x) = 1/f'(g(x))}$$

- $\log'(x) = 1/\exp'(\log(x)) = 1/\exp(\log(x)) = 1/x$.

The arcsin function

- Consider $y = \arcsin(x)$, so $x = \sin(y)$.

$$\frac{dx}{dy} = \sin'(y) = \cos(y)$$
$$\frac{dy}{dx} = 1/\frac{dx}{dy} = \cos(y)^{-1}.$$

- Also $\sin(y)^2 + \cos(y)^2 = 1$, so

$$\cos(y) = \sqrt{1 - \sin(y)^2} = \sqrt{1 - x^2}$$
$$\cos(y)^{-1} = (1 - x^2)^{-1/2}$$

- So $\arcsin'(x) = \frac{dy}{dx} = (1 - x^2)^{-1/2}$.

The arctanh function

- Consider $y = \operatorname{arctanh}(x)$, so $x = \tanh(y) = \frac{\sinh(y)}{\cosh(y)}$.

$$\begin{aligned}
\frac{dx}{dy} &= \tanh'(y) \\
&= \frac{\sinh'(y)\cosh(y) - \sinh(y)\cosh'(y)}{\cosh(y)^2} \\
&= \frac{\cosh(y)^2 - \sinh(y)^2}{\cosh(y)^2} \\
&= 1 - \tanh(y)^2 = 1 - x^2 \\
\frac{dy}{dx} &= 1\Big/\frac{dx}{dy} = \frac{1}{1-x^2}.
\end{aligned}$$

- So $\operatorname{arctanh}'(x) = \frac{dy}{dx} = (1-x^2)^{-1}$.

Classes of functions

- If $f(x)$ is a polynomial, then so is $f'(x)$.
  - Eg $f(x) = x + x^{10} + x^{100}$; $\quad f'(x) = 1 + 10x^9 + 100x^{99}$
  - Eg $f(x) = (x-1)^4 + (x+1)^4$; $\quad f'(x) = 4(x-1)^3 + 4(x+1)^3$
- If $f(x)$ is a rational function, then so is $f'(x)$.
  - Eg $f(x) = \frac{x^2-1}{x^2+1}$; $\quad f'(x) = \frac{4x}{(x^2+1)^2}$
  - Eg $f(x) = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2}$; $\quad f'(x) = -\frac{1}{x^2} - \frac{1}{(x+1)^2} - \frac{1}{(x+2)^2}$
- If $f(x)$ is a trigonometric polynomial, so is $f'(x)$.
  - Eg $f(x) = \sin(x) + \sin(3x)/3 + \sin(5x)/5$;
    $f'(x) = \cos(x) + \cos(3x) + \cos(5x)$.
  - Eg $f(x) = \sin(3x) + \cos(3x)$; $\quad f'(x) = 3\cos(3x) - 3\sin(3x)$.
- If $f(x)$ is a polynomial times e^x, so is $f'(x)$.
  - Eg $f(x) = (x + x^2)e^x$; $\quad f'(x) = (1 + 3x + x^2)e^x$.
  - Eg $f(x) = (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x$; $\quad f'(x) = x^4e^x$.

Implicit differentiation

- Suppose that x and y are related by an equation such as $y^4 + xy = x^3$. We cannot write y as a function of x, but we can still find dy/dx.
- Differentiate both sides. Terms in the equation involving y give terms in the derivative involving dy/dx. Rearranging gives dy/dx in terms of x and y.
- Suppose that $y^4 + xy = x^3$, so

$$\tfrac{d}{dx}\left(y^4 + xy\right) = \tfrac{d}{dx}\left(x^3\right) = 3x^2.$$

Also $\frac{d}{dx}(y^4) = 4y^3\frac{dy}{dx}$ by the power rule
and $\frac{d}{dx}(xy) = \frac{dx}{dx}y + x\frac{dy}{dx} = y + x\frac{dy}{dx}$ by the product rule ; so

$$\begin{aligned} 4y^3\tfrac{dy}{dx} + y + x\tfrac{dy}{dx} &= 3x^2 \\ (4y^3 + x)\tfrac{dy}{dx} &= 3x^2 - y \\ \tfrac{dy}{dx} &= \tfrac{3x^2 - y}{4y^3 + x}. \end{aligned}$$

Implicit examples

- Suppose $x + \sin(x) = y - \cos(y)$.

$$\begin{aligned} \tfrac{d}{dx}(x + \sin(x)) &= \tfrac{d}{dx}(y - \cos(y)) \\ 1 + \cos(x) &= \tfrac{dy}{dx} + \sin(y)\tfrac{dy}{dx} \\ \tfrac{dy}{dx} &= \tfrac{1+\cos(x)}{1+\sin(y)} \end{aligned}$$

- Suppose $y = \exp(x^2 + y^2)$.

$$\begin{aligned} \tfrac{dy}{dx} &= \tfrac{d}{dx}\exp(x^2 + y^2) = \tfrac{d}{dx}(e^{x^2}e^{y^2}) \\ &= 2xe^{x^2}e^{y^2} + e^{x^2}.2y\tfrac{dy}{dx}e^{y^2} \\ &= 2(x + y\tfrac{dy}{dx})\exp(x^2 + y^2) \\ (1 - 2y\exp(x^2 + y^2))\tfrac{dy}{dx} &= 2x\exp(x^2 + y^2) \\ \tfrac{dy}{dx} &= \tfrac{2x\exp(x^2+y^2)}{1-2y\exp(x^2+y^2)} \end{aligned}$$

Parametric differentiation

- Suppose that x and y are both functions of another variable t. Then

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

- Suppose that $x = 1 + t^2$ and $y = t + t^3$ (so $t = y/x$)

$$dy/dt = 1 + 3t^2 \qquad\qquad dx/dt = 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+3t^2}{2t} = \frac{1+3(y/x)^2}{2(y/x)} = \frac{x^2+3y^2}{2xy}$$

- Suppose that $x = t - \sin(t)$ and $y = 1 - \cos(t)$.

$$dy/dt = \sin(t) \qquad\qquad dx/dt = 1 - \cos(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin(t)}{1-\cos(t)} = \frac{\sqrt{y(2-y)}}{y} = \sqrt{\frac{2-y}{y}}$$

The circle

- Consider a point (x, y) on the unit circle, so $x^2 + y^2 = 1$.
- Differentiate $x^2 + y^2 = 1$; $2x + 2y\dfrac{dy}{dx} = 0$;

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

- Parametrically: $x = \cos(t)$, $y = \sin(t)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t)}{-\sin(t)} = -\frac{x}{y}$$

- Directly: $y = (1 - x^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-1/2}\frac{d}{dx}(1 - x^2) = \frac{1}{2}y^{-1}.(-2x) = -\frac{x}{y}.$$