

Differentiation

The chain rule

- ▶ Suppose that y depends on u , and u depends on x . Then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

- ▶ If x changes to $x + \delta x$, then u changes to $u + \delta u$ and y changes to $y + \delta y$. Clearly

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \frac{\delta u}{\delta x}.$$

In the limit, δx , δu and δy all approach zero, and we get

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

- ▶ Alternative notation: suppose that $f(x) = g(h(x))$. Then

$$f'(x) = g'(h(x))h'(x)$$

Examples of the chain rule

- ▶ Consider $y = \cos(x^2)$. This is $y = \cos(u)$, where $u = x^2$.

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = -\sin(u) = -\sin(x^2)$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -\sin(x^2) \cdot 2x = -2x \sin(x^2).$$

- ▶ Consider $f(x) = \exp(\sin(x))$.

$$f'(x) = \exp'(\sin(x)) \cdot \sin'(x) = \exp(\sin(x)) \cos(x).$$

- ▶ Consider $y = a \sin(bx + c)$. Put $u = bx + c$, so $y = a \sin(u)$.
Then $\frac{du}{dx} = b$ and $\frac{dy}{du} = a \cos(u)$ so

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = a \cos(u) \cdot b = ab \cos(u) = ab \cos(bx + c).$$

The power rule

- ▶ If u depends on x and n does not, then

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$

- ▶ Reason: If $y = u^n$ then $\frac{dy}{du} = nu^{n-1}$ so $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = nu^{n-1} \frac{du}{dx}$
- ▶ Consider $y = \sqrt{1+x^2}$. This is $y = u^{1/2}$, where $u = 1+x^2$. Then

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{1+x^2}} \quad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} 2x = \frac{x}{\sqrt{1+x^2}}$$

- ▶ $\frac{d}{dx}(\sin(x)^5) = 5 \sin(x)^4 \cos(x)$
- ▶ $\frac{d}{dx}(\log(x)^3) = 3 \log(x)^2 x^{-1} = 3 \log(x)^2 / x$

The logarithmic rule



$$\frac{d}{dx} \log(u) = \frac{1}{u} \frac{du}{dx}$$

$$\frac{du}{dx} = u \frac{d}{dx} \log(u)$$



$$\frac{d}{dx} \log(\cos(x)) = \frac{1}{\cos(x)} \cos'(x) = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$



$$\frac{d}{dx} \log(1+x^2) = \frac{\frac{d}{dx}(1+x^2)}{1+x^2} = \frac{2x}{1+x^2}$$

▶ Consider $y = x^x$, so $\log(y) = x \log(x)$. Then

$$\begin{aligned} \frac{d}{dx} \log(y) &= \frac{d}{dx} (x \log(x)) \\ &= 1 \cdot \log(x) + x \cdot x^{-1} = \log(x) + 1 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= y \frac{d}{dx} \log(y) \\ &= x^x (\log(x) + 1). \end{aligned}$$

The inverse function rule

- ▶ If x and y are interdependent variables, then

$$\frac{dx}{dy} = 1 / \frac{dy}{dx}$$

- ▶ (Take limits in the obvious relation $\frac{\delta x}{\delta y} = 1 / \frac{\delta y}{\delta x}$.)
- ▶ Consider $y = \log(x)$, so $x = e^y$.

$$\frac{dx}{dy} = e^y = x \qquad \frac{dy}{dx} = 1 / \frac{dx}{dy} = \frac{1}{x}$$

- ▶ Alternative notation: if $y = g(x)$ then $x = f(y)$, where $f = g^{-1}$ and $g = f^{-1}$. Then

$$g'(x) = 1 / f'(g(x))$$

- ▶ $\log'(x) = 1 / \exp'(\log(x)) = 1 / \exp(\log(x)) = 1/x$.

The arcsin function

- ▶ Consider $y = \arcsin(x)$, so $x = \sin(y)$.

$$\frac{dx}{dy} = \sin'(y) = \cos(y)$$

$$\frac{dy}{dx} = 1/\frac{dx}{dy} = \cos(y)^{-1}.$$

- ▶ Also $\sin(y)^2 + \cos(y)^2 = 1$, so

$$\cos(y) = \sqrt{1 - \sin(y)^2} = \sqrt{1 - x^2}$$

$$\cos(y)^{-1} = (1 - x^2)^{-1/2}$$

- ▶ So $\arcsin'(x) = \frac{dy}{dx} = (1 - x^2)^{-1/2}$.

The arctanh function

- ▶ Consider $y = \operatorname{arctanh}(x)$, so $x = \tanh(y) = \frac{\sinh(y)}{\cosh(y)}$.

$$\begin{aligned}\frac{dx}{dy} &= \tanh'(y) \\ &= \frac{\sinh'(y) \cosh(y) - \sinh(y) \cosh'(y)}{\cosh(y)^2} \\ &= \frac{\cosh(y)^2 - \sinh(y)^2}{\cosh(y)^2} \\ &= 1 - \tanh(y)^2 = 1 - x^2 \\ \frac{dy}{dx} &= 1 / \frac{dx}{dy} = \frac{1}{1 - x^2}.\end{aligned}$$

- ▶ So $\operatorname{arctanh}'(x) = \frac{dy}{dx} = (1 - x^2)^{-1}$.

- ▶ If $f(x)$ is a polynomial, then so is $f'(x)$.
 - ▶ Eg $f(x) = x + x^{10} + x^{100}$; $f'(x) = 1 + 10x^9 + 100x^{99}$
 - ▶ Eg $f(x) = (x - 1)^4 + (x + 1)^4$; $f'(x) = 4(x - 1)^3 + 4(x + 1)^3$
- ▶ If $f(x)$ is a rational function, then so is $f'(x)$.
 - ▶ Eg $f(x) = \frac{x^2-1}{x^2+1}$; $f'(x) = \frac{4x}{(x^2+1)^2}$
 - ▶ Eg $f(x) = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2}$; $f'(x) = -\frac{1}{x^2} - \frac{1}{(x+1)^2} - \frac{1}{(x+2)^2}$
- ▶ If $f(x)$ is a trigonometric polynomial, so is $f'(x)$.
 - ▶ Eg $f(x) = \sin(x) + \sin(3x)/3 + \sin(5x)/5$;
 $f'(x) = \cos(x) + \cos(3x) + \cos(5x)$.
 - ▶ Eg $f(x) = \sin(3x) + \cos(3x)$; $f'(x) = 3\cos(3x) - 3\sin(3x)$.
- ▶ If $f(x)$ is a polynomial times e^x , so is $f'(x)$.
 - ▶ Eg $f(x) = (x + x^2)e^x$; $f'(x) = (1 + 3x + x^2)e^x$.
 - ▶ Eg $f(x) = (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x$; $f'(x) = x^4e^x$.

Implicit differentiation

- ▶ Suppose that x and y are related by an equation such as $y^4 + xy = x^3$. We cannot write y as a function of x , but we can still find dy/dx .
- ▶ Differentiate both sides. Terms in the equation involving y give terms in the derivative involving dy/dx . Rearranging gives dy/dx in terms of x and y .
- ▶ Suppose that $y^4 + xy = x^3$, so

$$\frac{d}{dx} (y^4 + xy) = \frac{d}{dx} (x^3) = 3x^2.$$

Also $\frac{d}{dx}(y^4) = 4y^3 \frac{dy}{dx}$ by the power rule
and $\frac{d}{dx}(xy) = \frac{dx}{dx}y + x\frac{dy}{dx} = y + x\frac{dy}{dx}$ by the product rule ; so

$$\begin{aligned}4y^3 \frac{dy}{dx} + y + x \frac{dy}{dx} &= 3x^2 \\(4y^3 + x) \frac{dy}{dx} &= 3x^2 - y \\ \frac{dy}{dx} &= \frac{3x^2 - y}{4y^3 + x}.\end{aligned}$$

- ▶ Suppose $x + \sin(x) = y - \cos(y)$.

$$\frac{d}{dx}(x + \sin(x)) = \frac{d}{dx}(y - \cos(y))$$

$$1 + \cos(x) = \frac{dy}{dx} + \sin(y) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1 + \cos(x)}{1 + \sin(y)}$$

- ▶ Suppose $y = \exp(x^2 + y^2)$.

$$\frac{dy}{dx} = \frac{d}{dx} \exp(x^2 + y^2) = \frac{d}{dx}(e^{x^2} e^{y^2})$$

$$= 2xe^{x^2} e^{y^2} + e^{x^2} \cdot 2y \frac{dy}{dx} e^{y^2}$$

$$= 2(x + y \frac{dy}{dx}) \exp(x^2 + y^2)$$

$$(1 - 2y \exp(x^2 + y^2)) \frac{dy}{dx} = 2x \exp(x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{2x \exp(x^2 + y^2)}{1 - 2y \exp(x^2 + y^2)}$$

Parametric differentiation

- Suppose that x and y are both functions of another variable t . Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- Suppose that $x = 1 + t^2$ and $y = t + t^3$ (so $t = y/x$)

$$dy/dt = 1 + 3t^2 \qquad dx/dt = 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + 3t^2}{2t} = \frac{1 + 3(y/x)^2}{2(y/x)} = \frac{x^2 + 3y^2}{2xy}$$

- Suppose that $x = t - \sin(t)$ and $y = 1 - \cos(t)$.

$$dy/dt = \sin(t) \qquad dx/dt = 1 - \cos(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin(t)}{1 - \cos(t)} = \frac{\sqrt{y(2-y)}}{y} = \sqrt{\frac{2-y}{y}}$$

The circle

- ▶ Consider a point (x, y) on the unit circle, so $x^2 + y^2 = 1$.
- ▶ Differentiate $x^2 + y^2 = 1$; $2x + 2y \frac{dy}{dx} = 0$;

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

- ▶ Parametrically: $x = \cos(t)$, $y = \sin(t)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t)}{-\sin(t)} = -\frac{x}{y}$$

- ▶ Directly: $y = (1 - x^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-1/2} \frac{d}{dx}(1 - x^2) = \frac{1}{2}y^{-1} \cdot (-2x) = -\frac{x}{y}.$$