Integration

Introduction

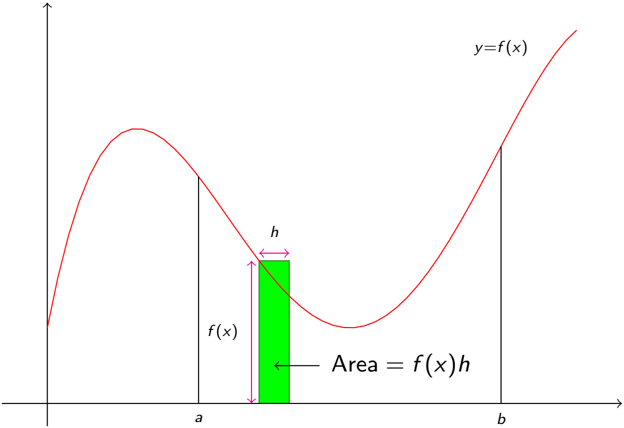
Things you should know:

- The meaning of integration (take the sum of a large number of very small contributions, and pass to the limit)
- Integration as the reverse of differentiation
- Integrals of standard functions and classes of functions
- The method of undetermined coefficients
- Integration by parts
- Integration by substitution

Meaning

- To define $\int_a^b f(x)\,dx$:
  - Divide the interval $[a, b]$ into many short intervals $[x, x+h]$.
  - For each short interval $[x, x+h]$, find $f(x)h$.
  - Add these terms together to get an approximation to $\int_a^b f(x)\,dx$.
  - For the exact value of $\int_a^b f(x)\,dx$, take the limit $h \to 0$.
- In economics, government revenue depends on time, and total revenue in the last decade is $\int_{1999}^{2009} \text{revenue(t)}\,dt$.
- If a particle moves with velocity $v(t) > 0$ at time t, then the total distance moved between times a and b is $\int_a^b v(t)\,dt$.
- A current flowing in a wire exerts a magnetic force on a moving electron. There is a formula for the force contributed by a short section of wire; to get the total force, we integrate.

Areas

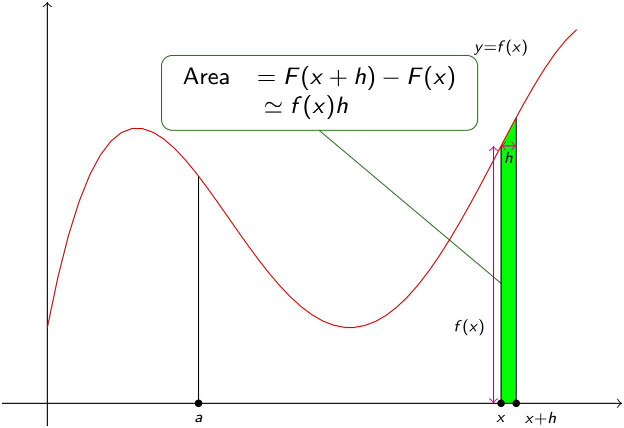


Consider the integral $\int_a^b f(x)\,dx$.For each short interval $[x, x+h] \subset [a, b]$, we have a contribution $f(x)h$. This is the area of the green rectangle.This is the contribution from one short interval, but we need to add together the contributions from many short intervals.

The Fundamental Theorem of Calculus

- An *indefinite integral* of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.
- Examples:
  - $\log(x)$ is an indefinite integral of $1/x$
  - $\sin(x)$ is an indefinite integral of $\cos(x)$
  - $F(x) = x^2 + 2x$ and $G(x) = (x+1)^2$ are indefinite integrals of $2x + 2$
- The *Fundamental Theorem of Calculus:*
  - For any number a, the function $F(x) = \int_a^x f(t)\,dt$ is an indefinite integral of $f(x)$.
  - If $F(x)$ is any indefinite integral of $f(x)$, then $\int_a^b f(x)\,dx = \big[F(x)\big]_a^b = F(b) - F(a)$.
- The functions $F(x) = \int_0^x 2t + 2\,dt = x^2 + 2x$ and $G(x) = \int_{-1}^x 2t + 2\,dt = (x+1)^2$ are both indefinite integrals of $2x + 2$.
- $\int_a^b \frac{1}{x} = \big[\log(x)\big]_a^b = \log(b) - \log(a)$

Proof of the Fundamental Theorem



$$F'(x) = \lim_{h \longrightarrow 0} (F(x+h) - F(x))/h = f(x).$$

Constants

- Is it $\int x^2\,dx = x^3/3$ or $\int x^2\,dx = x^3/3 + c$?
- Either is acceptable in the exam.
  Neither one is strictly logically satisfactory.
- $x^3/3$ is *an* indefinite integral of x^2.
- *Every* indefinite integral of x^2 has the form $x^3/3 + c$ for some c.
- If you just want to calculate $\int_a^b f(x)\,dx$, it does not matter which indefinite integral you use. Any two choices will give the same answer.
- In solving differential equations, it often does matter which indefinite integral you use. You must therefore include a '+c' term, and do some extra work to see what c should be.
- Maple's `int()` command will never give you a '+c' term.
  If you need one, you must insert it yourself.

Checking and Guessing

- Integrals can easily be checked by differentiating

- $\int \sin(x)^2\,dx = \sin(x)^3/3$? $\int \sin(x)^2\,dx \neq \sin(x)^3/3$, because

$$\frac{d}{dx}\left(\sin(x)^3/3\right) = 3\sin(x)^2\cos(x)/3 = \sin(x)^2\cos(x) \neq \sin(x)^2.$$

- $\int \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\,dx = \frac{\sin(x)}{x}$? $\int \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}\,dx = \frac{\sin(x)}{x}$, because

$$\frac{d}{dx}\left(\frac{\sin(x)}{x}\right) = \frac{\sin'(x).x - \sin(x).1}{x^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}.$$

- $\int 2x\,e^{x^2}\,dx = e^{x^2}$, because $\frac{d}{dx}e^{x^2} = 2x\,e^{x^2}$.
- $\int \frac{3x^2+2x+1}{x^3+x^2+x+1}\,dx = \log(x^3+x^2+x+1)$, because

$$\frac{d}{dx}\log(x^3+x^2+x+1) = \frac{\frac{d}{dx}(x^3+x^2+x+1)}{x^3+x^2+x+1} = \frac{3x^2+2x+1}{x^3+x^2+x+1}.$$

Undetermined coefficients

- Suppose we know that for some constants $a, \ldots, d$
$$\int \log(x)^3 \, dx = (a \log(x)^3 + b \log(x)^2 + c \log(x) + d)x$$
(How could we know this? — see later)
- **Problem:** find a, b, c and d.
- $$\begin{aligned} \log(x)^3 &= \tfrac{d}{dx}\left((a \log(x)^3 + b \log(x)^2 + c \log(x) + d)x\right) \\ &= (3a \log(x)^2 x^{-1} + 2b \log(x) x^{-1} + cx^{-1})x + \\ &\quad (a \log(x)^3 + b \log(x)^2 + c \log(x) + d).1 \\ &= a \log(x)^3 + (b + 3a) \log(x)^2 + (c + 2b) \log(x) + (d + c) \end{aligned}$$
- So $a = 1$, $b + 3a = 0$, $c + 2b = 0$ and $d + c = 0$ (compare coefficients)
- So $a = 1$, $b = -3$, $c = 6$ and $d = -6$
$$\int \log(x)^3 \, dx = (\log(x)^3 - 3 \log(x)^2 + 6 \log(x) - 6)x.$$

Standard integrals

$$\begin{aligned}
\int \exp(x)\,dx &= \exp(x) \\
\int \cosh(x)\,dx &= \sinh(x) \\
\int \sinh(x)\,dx &= \cosh(x) \\
\int \operatorname{sech}(x)^2\,dx &= \tanh(x) \\
\int \cos(x)\,dx &= \sin(x) \\
\int \sin(x)\,dx &= -\cos(x) \\
\int \sec(x)^2\,dx &= \tan(x)
\end{aligned}$$

$$\begin{aligned}
\int 1/x\,dx &= \log(x) \\
\int (1+x^2)^{-1/2}\,dx &= \operatorname{arcsinh}(x) \\
\int (x^2-1)^{-1/2}\,dx &= \operatorname{arccosh}(x) \\
\int (1-x^2)^{-1}\,dx &= \operatorname{arctanh}(x) \\
\int (1-x^2)^{-1/2}\,dx &= \arcsin(x) \\
\int (1-x^2)^{-1/2}\,dx &= -\arccos(x) \\
\int (1+x^2)^{-1}\,dx &= \arctan(x)
\end{aligned}$$

$$\begin{aligned}
\int x^n\,dx &= x^{n+1}/(n+1) \qquad (n \neq -1) \\
\int a^x\,dx &= a^x/\log(a) \\
\int \log(x)\,dx &= x\log(x) - x \\
\int \tan(x)\,dx &= -\log(\cos(x)) \\
\int \sin(x)^2\,dx &= (2x - \sin(2x))/4 \\
\int \cos(x)^2\,dx &= (2x + \sin(2x))/4
\end{aligned}$$

Rational functions

- A *rational function* of x is a function defined using only constants, addition, multiplication, division and integer powers.
- No roots, fractional powers, logs, exponentials, trigonometric functions and so on can occur in a rational function.
- **Examples:** $\frac{1+x+x^2}{1-x+x^2}$ $\quad \frac{1}{x}+\frac{\pi}{x-1}+\frac{\pi^2}{x-2}$ $\quad x^2+x+1+x^{-1}+x^{-2}$
- **Non-Examples:** $e^{-x}\sin(x)$ $\quad \sqrt{1-x^2}$ $\quad \frac{\log(x)}{1+x}$ $\quad \frac{\arctan(x)}{2\pi}$.
- If $f(x)$ is a rational function, then $\int f(x)\,dx$ is a sum of terms of the following types:
  - Rational functions
  - Terms of the form $\ln(|x-u|)$
  - Terms of the form $\ln(x^2+vx+w)$
  - Terms of the form $\arctan(ux+v)$.
- $\int \frac{4x^3+8}{x^6-x^2}\,dx = \frac{8}{x}+3\ln(|x-1|)-\ln(|x+1|)-\ln(x^2+1)+4\arctan(x)$

Rational function examples

- $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx = x + \ln(|x-1|) + \ln(|x+1|)$
- $\displaystyle\int \left(\frac{x+1}{x-1}\right)^3 dx = 1 + \frac{6}{x-1} + \frac{12}{(x-1)^2} + \frac{8}{(x-1)^3}$
- $\displaystyle\int \frac{2x+2}{x^2+1}\,dx = \ln(x^2+1) + 2\arctan(x)$
- $\displaystyle\int \frac{1}{x^{-1}+1+x}\,dx = \frac{1}{2}\ln(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{1+2x}{\sqrt{3}}\right)$
- $\displaystyle\int \frac{4}{1-x^4}\,dx = \ln(|x+1|) - \ln(|x-1|) + 2\arctan(x)$

$$\frac{d}{dx}\ln(|x-u|) = \frac{1}{x-u} \qquad \frac{d}{dx}\ln(x^2+ux+v) = \frac{2x+u}{x^2+ux+v}$$

$$\frac{d}{dx}\arctan(ux+v) = \frac{u}{1+(ux+v)^2} = \frac{u}{u^2x^2+2uvx+(v^2+1)}$$

Trigonometric polynomials

$$\int \sin(nx)\,dx = -\cos(nx)/n \qquad \int \cos(nx)\,dx = \sin(nx)/n$$

$$\begin{aligned}
\cos(2x) &= \cos(x)^2 - \sin(x)^2 = 2\cos(x)^2 - 1 = 1 - 2\sin(x)^2 \\
\sin(x)^2 &= 1/2 - \cos(2x)/2 \\
\int \sin(x)^2\,dx &= x/2 - \sin(2x)/4 \\
\int \cos(x)^2\,dx &= x/2 + \sin(2x)/4 \\
\sin(x)^3 &= 3\sin(x)/4 - \sin(3x)/4 \\
\int \sin(x)^3\,dx &= -3\cos(x)/4 + \cos(3x)/12 \\
\sin(x)\sin(2x)\sin(4x) &= -\sin(x)/4 + \sin(3x)/4 + \sin(5x)/4 - \sin(7x)/4 \\
\int \sin(x)\sin(2x)\sin(4x)\,dx &= \cos(x)/4 - \cos(3x)/12 - \cos(5x)/20 + \cos(7x)/28 \\
\sin(x)^4 + \cos(x)^4 &= 3/4 + \cos(4x)/4 \\
\int \sin(x)^4 + \cos(x)^4\,dx &= 3x/4 + \sin(4x)/16
\end{aligned}$$