Integration 2

Affine substitution

If $\int f(x)\,dx = g(x)$ and a, b are constant, then

$$\int f(ax+b)\,dx = g(ax+b)/a$$

$$\int \cos(x)\,dx = \sin(x) \qquad \int \cos(2x+3)\,dx = \sin(2x+3)/2$$

$$\int e^x\,dx = e^x \qquad \int e^{-2x+7}\,dx = e^{-2x+7}/(-2)$$

$$\int \tan(x)\,dx = -\ln(\cos(x)) \qquad \int \tan(\pi x)\,dx = -\ln(\cos(\pi x))/\pi$$

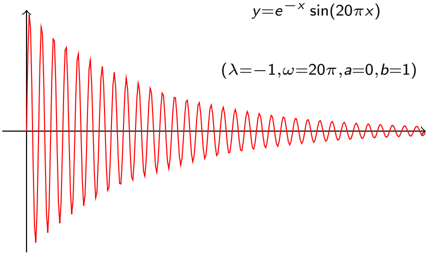
Exponential oscillations

- An *exponential oscillation* is a function of the form

$$f(x) = e^{\lambda x}(a\cos(\omega x) + b\sin(\omega x)),$$

where a, b, λ and ω are constants.

- The *growth rate* is λ, and the *angular frequency* is ω.



- Special cases:

$$\begin{aligned}
f(x) &= e^{\lambda x}\sin(\omega x) && (a = 0, b = 1)\\
f(x) &= a\cos(\omega x) + b\sin(\omega x) && (\lambda = 0)\\
f(x) &= ae^{\lambda x} && (\omega = 0).
\end{aligned}$$

Integrating exponential oscillations

The integral of an EO is another EO with the same growth rate and angular frequency.

$$\boxed{\int e^{\lambda x}(a\cos(\omega x) + b\sin(\omega x))\,dx = e^{\lambda x}(A\cos(\omega x) + B\sin(\omega x))}$$

$$\boxed{A = \frac{a\lambda - b\omega}{\lambda^2 + \omega^2} \qquad B = \frac{a\omega + b\lambda}{\lambda^2 + \omega^2}.}$$

- Example: find $\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx \int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx$
  - $\lambda = -2$, $\omega = 4$, $a = 5$, $b = -3$
  - $A = \dfrac{a\lambda - b\omega}{\lambda^2 + \omega^2} = \dfrac{5.(-2) - (-3).4}{(-2)^2 + 4^2} = 1/10$
  - $B = \dfrac{a\omega + b\lambda}{\lambda^2 + \omega^2} = \dfrac{5.4 + (-3)(-2)}{(-2)^2 + 4^2} = 13/10$

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(\cos(4x) + 13\sin(4x))/10$$

Integrating exponential oscillations

Alternatively:

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(A\cos(4x) + B\sin(4x)) \text{ for some } A, B$$

$$\begin{aligned}
e^{-2x}(5\cos(4x) - 3\sin(4x)) &= \frac{d}{dx}\left(e^{-2x}(A\cos(4x) + B\sin(4x))\right) \\
&= -2e^{-2x}(A\cos(4x) + B\sin(4x)) + \\
&\quad e^{-2x}(-4A\sin(4x) + 4B\cos(4x)) \\
&= e^{-2x}((4B - 2A)\cos(4x) - (2B + 4A)\sin(4x))
\end{aligned}$$

By comparing coefficients, we must have $4B - 2A = 5$ and $2B + 4A = 3$. These equations can be solved to give $A = 1/10$ and $B = 13/10$. Thus

$$\int e^{-2x}(5\cos(4x) - 3\sin(4x))\,dx = e^{-2x}(\cos(4x) + 13\sin(4x))/10.$$

Polynomial exponential oscillations

- A *polynomial exponential oscillation* is a function of the form

$$f(x) = e^{\lambda x}(a(x)\cos(\omega x) + b(x)\sin(\omega x)),$$

where $a(x)$ and $b(x)$ are polynomials.

- λ is the *growth rate* and ω is the *angular frequency*. The *degree* is the highest power of x that occurs in $a(x)$ or in $b(x)$.
- The function $f(x) = e^{-2x}((1 + x^5)\cos(4x) + x^3\sin(4x))$ is a PEO of growth rate -2, frequency 4 and degree 5.
- The function $f(x) = e^{4x}((1 + x^3 + x^6)\sin(3x))$ is a PEO of growth rate 4, frequency 3 and degree 6.
- **Fact:** The integral of any PEO is another PEO with the same growth rate, frequency and degree.

Integrating PEO's — I

- $\int xe^{-x}\sin(x)\,dx$ is a PEO of degree 1, growth -1, frequency 1.
- $\int xe^{-x}\sin(x)\,dx = (Ax+B)e^{-x}\cos(x) + (Cx+D)e^{-x}\sin(x)$ for some A, B, C, D.
- $$\begin{aligned} xe^{-x}\sin(x) &= \tfrac{d}{dx}\left((Ax+B)e^{-x}\cos(x) + (Cx+D)e^{-x}\sin(x)\right) \\ &= Ae^{-x}\cos(x) - (Ax+B)e^{-x}\cos(x) - (Ax+B)e^{-x}\sin(x) + \\ &\quad Ce^{-x}\sin(x) - (Cx+D)e^{-x}\sin(x) + (Cx+D)e^{-x}\cos(x) \\ &= (-A+C)xe^{-x}\cos(x) + (A-B+D)e^{-x}\cos(x) + \\ &\quad (-A-C)xe^{-x}\sin(x) + (-B+C-D)e^{-x}\sin(x). \end{aligned}$$

- $-A+C=0$, $A-B+D=0$, $-A-C=1$, $-B+C-D=0$.
- So $A=-1/2$, $B=-1/2$, $C=-1/2$, $D=0$
- $\int xe^{-x}\sin(x)\,dx = -((x+1)e^{-x}\cos(x) + xe^{-x}\sin(x))/2$.

Integrating PEO's — II

- $\int x^3 e^x \, dx$ is a PEO of degree 3, growth 1 and frequency 0.
- $\int x^3 e^x \, dx = (Ax^3 + Bx^2 + Cx + D)e^x$ for some A, B, C, D.
- $$\begin{aligned} x^3 e^x &= \frac{d}{dx}\left((Ax^3 + Bx^2 + Cx + D)e^x\right) \\ &= (3Ax^2 + 2Bx + C)e^x + (Ax^3 + Bx^2 + Cx + D)e^x \\ &= (Ax^3 + (3A + B)x^2 + (2B + C)x + (C + D))e^x. \end{aligned}$$
- $A = 1$, $3A + B = 0$, $2B + C = 0$, $C + D = 0$.
- so $A = 1$, $B = -3$, $C = 6$, $D = -6$
- so $\int x^3 e^x \, dx = (x^3 - 3x^2 + 6x - 6)e^x$.

Integration by parts — I

- Consider $\int xe^{x/a}\,dx$.
- Consider $\int xe^{x/a}\,dx$.
- $u = x$ $\qquad dv/dx = e^{x/a}$
- $du/dx = 1$ $\qquad v = a\,e^{x/a}$
- $\int xe^{x/a}\,dx = uv - \int \frac{du}{dx} v\,dx = axe^{x/a} - \int a\,e^{x/a}\,dx = axe^{x/a} - a^2 e^{x/a}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx}$$

- This is most useful when (a) du/dx is simpler than u (eg u polynomial) and (b) v is no more complicated than dv/dx (eg $dv/dx = \cos(x)$).

Integration by parts — II

- Consider $\int (1 - \ln(x))x^{-2}\, dx$.
- Consider $\int (1 - \ln(x))x^{-2}\, dx$.
- $u = 1 - \ln(x)$ $\quad\quad dv/dx = x^{-2}$
- $du/dx = -x^{-1}$ $\quad\quad v = -x^{-1}$
- $\int (1 - \ln(x))x^{-2}\, dx = uv - \int \frac{du}{dx} v\, dx = -(1 - \ln(x))x^{-1} - \int x^{-2}\, dx$

  $= (\ln(x) - 1)x^{-1} + x^{-1} = \ln(x)/x$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula: $$\boxed{\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx}$$

Integration by parts — III

- Consider $\displaystyle\int x\sin(\omega x)\,dx$.
- Consider $\displaystyle\int x\sin(\omega x)\,dx$.
- $u = x$ $\qquad$ $dv/dx = \sin(\omega x)$
- $du/dx = 1$ $\qquad$ $v = -\omega^{-1}\cos(\omega x)$
- $\displaystyle\int x\sin(\omega x)\,dx = uv - \int \frac{du}{dx} v\,dx = -\omega^{-1}x\cos(\omega x) + \int \omega^{-1}\cos(\omega x)\,dx$

  $= -\omega^{-1}x\cos(\omega x) + \omega^{-2}\sin(\omega x)$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx}$$

Integration by parts — IV

- Consider $\int \arcsin(x)\,dx$.
- Consider $\int \arcsin(x).1\,dx$.
- $u = \arcsin(x)$ $\qquad$ $dv/dx = 1$
- $du/dx = (1-x^2)^{-1/2}$ $\qquad$ $v = x$
- $\int \arcsin(x).1\,dx = uv - \int \frac{du}{dx} v\,dx = \arcsin(x).x - \int x(1-x^2)^{-1/2}\,dx$

  $= x\arcsin(x) + (1-x^2)^{1/2}$

- To integrate a product, call the factors u and $\frac{dv}{dx}$.
- Differentiate u to find du/dx.
- Integrate $\frac{dv}{dx}$ to find v.
- Use the formula:

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v\,dx}$$