Integration 3

Integration by substitution — I

- Consider $\displaystyle\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Consider $\displaystyle\int \frac{\sin(x)}{\cos(x)^n}\,dx$.
- Put $u=\cos(x)$, so $du/dx=-\sin(x)$, so $dx=-du/\sin(x)$
- $$\begin{aligned}\int \frac{\sin(x)}{\cos(x)^n}\,dx &= \int \frac{\sin(x)}{u^n}\,\frac{-du}{\sin(x)} = -\int u^{-n}\,du \\ &= u^{1-n}/(n-1) = \frac{\cos(x)^{1-n}}{n-1}\end{aligned}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — II

- Consider $\displaystyle\int xe^{-4x^2}\,dx$.
- Consider $\displaystyle\int xe^{-4x^2}\,dx$.
- Put $u = -4x^2$, so $du/dx = -8x$, so $dx = -du/(8x)$
- $$\begin{aligned}\int xe^{-4x^2}\,dx &= \int -xe^u \frac{du}{8x} = -\frac{1}{8}\int e^u\,du \\ &= -e^u/8 = -e^{-4x^2}/8\end{aligned}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — III

- Consider $\displaystyle\int \frac{dx}{4x^2+4x+2}$.
- Consider $\displaystyle\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1}$.
- Put $u = 2x+1$, so $du/dx = 2$, so $dx = du/2$
- $$\begin{aligned}\int \frac{dx}{4x^2+4x+2} &= \int \frac{du/2}{u^2+1} \\ &= \arctan(u)/2 = \arctan(2x+1)/2\end{aligned}$$

- To find $\int f(x)\,dx$, pick out some part of $f(x)$ and call it u.
- Find du/dx, and rearrange to express dx in terms of x and du.
- Rewrite the integral in terms of u and du.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — IV

- Consider $\displaystyle\int \frac{dx}{\sqrt{x-x^2}}$.
- Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t\,dt$

$$\sqrt{x-x^2} = \sqrt{t^2-t^4} = t\sqrt{1-t^2}$$

$$\int \frac{dx}{\sqrt{x-x^2}} = \int \frac{2t\,dt}{t\sqrt{1-t^2}} = 2\int \frac{dt}{\sqrt{1-t^2}}$$

$$= 2\arcsin(t) = 2\arcsin(\sqrt{x})$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.
- Evaluate the integral, then rewrite the result in terms of x.

Integration by substitution — V

- Consider $\displaystyle\int \log(x)^2\,dx$.
- Put $x = e^t$, so $dx/dt = e^t$, so $dx = e^t\,dt$

$$\begin{aligned}\int \log(x)^2\,dx = \int \log(e^t)^2 e^t\,dt = \int t^2 e^t\,dt \\ = (t^2 - 2t + 2)e^t = (\log(x)^2 - 2\log(x) + 2)x\end{aligned}$$

- To find $\int f(x)\,dx$, put x equal to some function of t.
- Find dx/dt, and rearrange to express dx in terms of t and dt.
- Rewrite the integral in terms of t and dt.
- Evaluate the integral, then rewrite the result in terms of x.

Examples I

- $\displaystyle\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{\cos'(x)}{\cos(x)}\,dx = -\log(\cos(x))$.
- Consider $\int x^2\tan(x^3)\,dx$. Put $u = x^3$, so $du = 3x^2\,dx$, so $dx = du/(3x^2)$.

$$\begin{aligned}\int x^2\tan(x^3)\,dx &= \int x^2\tan(u)\frac{du}{3x^2} = \frac{1}{3}\int \tan(u)\,du = -\log(\cos(u))/3 \\ &= -\log(\cos(x^3))/3\end{aligned}$$

- Consider $\int xe^{\sqrt{x}}\,dx$. Put $t = \sqrt{x}$, so $x = t^2$, so $dx = 2t\,dt$.

$$\begin{aligned}\int xe^{\sqrt{x}}\,dx &= \int t^2e^t.2t\,dt = 2\int t^3e^t\,dt = 2(t^3 - 3t^2 + 6t - 6)e^t \\ &= (2x^{3/2} - 6x + 12x^{1/2} - 12)e^{\sqrt{x}}\end{aligned}$$

Examples II

- $$\begin{aligned} \int (2(x^2+1)e^x)^2\,dx &= \int (4x^4+8x^2+4)e^{2x}\,dx \\ &= (Ax^4+Bx^3+Cx^2+Dx+E)e^{2x} \\ (4x^4+8x^2+4)e^{2x} &= \frac{d}{dx}((Ax^4+Bx^3+Cx^2+Dx+E)e^{2x}) \\ &= (4Ax^3+3Bx^2+2Cx+D)e^{2x}+ \\ &\quad (Ax^4+Bx^3+Cx^2+Dx+E).2e^{2x} \\ &= e^{2x}(2Ax^4+(4A+2B)x^3+(3B+2C)x^2+ \\ &\qquad (2C+2D)x+(D+2E)) \end{aligned}$$

  So $4=2A$, $0=4A+2B$, $8=3B+2C$, $0=2C+2D$, $4=D+2E$
  So $A=2,\ B=-4,\ C=10,\ D=-10,\ E=7$

  $$\int (2(x^2+1)e^x)^2\,dx = (2x^4-4x^3+10x^2-10x+7)e^{2x}.$$

Examples III

- $$\begin{aligned}\int 1+\cosh(x)+\cosh(x)^2\,dx &= \int 1+\frac{e^x+e^{-x}}{2}+\left(\frac{e^x+e^{-x}}{2}\right)^2\,dx\\ &= \frac{1}{4}\int 4+2e^x+2e^{-x}+e^{2x}+2+e^{-2x}\,dx\\ &= \frac{1}{4}\left(6x+2e^x-2e^{-x}+\tfrac{1}{2}e^{2x}-\tfrac{1}{2}e^{-2x}\right)\\ &= \frac{3}{2}x+\frac{e^x-e^{-x}}{2}+\frac{1}{4}\frac{e^{2x}-e^{-2x}}{2}\\ &= \frac{3}{2}x+\sinh(x)+\frac{1}{4}\sinh(2x).\end{aligned}$$

Examples IV

- To show that $\int \frac{dx}{\cos(x)} = \log\left(\frac{1+\sin(x)}{\cos(x)}\right)$:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1+\sin(x)}{\cos(x)}\right) &= \frac{\cos(x).\cos(x) - (1+\sin(x))(-\sin(x))}{\cos(x)^2} \\
&= \frac{\cos(x)^2 + \sin(x)^2 + \sin(x)}{\cos(x)^2} = \frac{1+\sin(x)}{\cos(x)^2} \\
\frac{d}{dx}\log\left(\frac{1+\sin(x)}{\cos(x)}\right) &= \left(\frac{1+\sin(x)}{\cos(x)}\right)^{-1} \frac{d}{dx}\left(\frac{1+\sin(x)}{\cos(x)}\right) \\
&= \frac{\cos(x)}{1+\sin(x)}\frac{1+\sin(x)}{\cos(x)^2} = \frac{1}{\cos(x)}
\end{aligned}$$

Examples V

- $$\begin{aligned}\int 8x\sin(x)\cos(x)\,dx &= \int 4x\sin(2x)\,dx \\ &= -2x\cos(2x) + \int 2\cos(2x)\,dx \\ &= -2x\cos(2x) + \sin(2x).\end{aligned}$$

- Consider $\int 10e^{-x}\sin(x)^2\,dx = \int 5e^{-x}\,dx + \int -5e^{-x}\cos(2x)\,dx$.

$$\int -5e^{-x}\cos(2x)\,dx = e^{-x}(A\cos(2x) + B\sin(2x))$$

$$-5e^{-x}\cos(2x) = e^{-x}((2B-A)\cos(2x) - (2A+B)\sin(2x))$$

$$A = 1, \qquad B = -2$$

$$\int 10e^{-x}\sin(x)^2\,dx = -5e^{-x} + e^{-x}\cos(2x) - 2e^{-x}\sin(2x).$$