

Integration 3

- ▶ Consider $\int \frac{\sin(x)}{\cos(x)^n} dx$.
- ▶ Consider $\int \frac{\sin(x)}{\cos(x)^n} dx$.
- ▶ Put $u = \cos(x)$, so $du/dx = -\sin(x)$, so $dx = -du/\sin(x)$
- ▶
$$\int \frac{\sin(x)}{\cos(x)^n} dx = \int \frac{\sin(x)}{u^n} \frac{-du}{\sin(x)} = - \int u^{-n} du$$
$$= u^{1-n}/(n-1) = \frac{\cos(x)^{1-n}}{n-1}$$

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- ▶ To find $\int f(x) dx$, pick out some part of $f(x)$ and call it u .
 - ▶ Find du/dx , and rearrange to express dx in terms of x and du .
 - ▶ Rewrite the integral in terms of u and du .
 - ▶ Evaluate the integral, then rewrite the result in terms of x .

▶ Consider $\int xe^{-4x^2} dx$.

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▶ Put $u = -4x^2$, so $du/dx = -8x$, so $dx = -du/(8x)$

▶
$$\begin{aligned}\int xe^{-4x^2} dx &= \int -xe^u \frac{du}{8x} = -\frac{1}{8} \int e^u du \\ &= -e^u/8 = -e^{-4x^2}/8\end{aligned}$$

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- ▶ Consider $\int \frac{dx}{4x^2 + 4x + 2}$.
- ▶ Consider $\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x + 1)^2 + 1}$.
- ▶ Put $u = 2x + 1$, so $du/dx = 2$, so $dx = du/2$
- ▶
$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{du/2}{u^2 + 1}$$
$$= \arctan(u)/2 = \arctan(2x + 1)/2$$

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 - ▶ Rewrite the integral in terms of u and du .
 - ▶ Evaluate the integral, then rewrite the result in terms of x .

- ▶ Consider $\int \frac{dx}{\sqrt{x-x^2}}$.
- ▶ Put $x = t^2$, so $dx/dt = 2t$, so $dx = 2t dt$

$$\begin{aligned}\sqrt{x-x^2} &= \sqrt{t^2-t^4} = t\sqrt{1-t^2} \\ \int \frac{dx}{\sqrt{x-x^2}} &= \int \frac{2t dt}{t\sqrt{1-t^2}} = 2 \int \frac{dt}{\sqrt{1-t^2}} \\ &= 2 \arcsin(t) = 2 \arcsin(\sqrt{x})\end{aligned}$$

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- ▶ To find $\int f(x) dx$, put x equal to some function of t .
 - ▶ Find dx/dt , and rearrange to express dx in terms of t and dt .
 - ▶ Rewrite the integral in terms of t and dt .
 - ▶ Evaluate the integral, then rewrite the result in terms of x .

- ▶ Consider $\int \log(x)^2 dx$.
- ▶ Put $x = e^t$, so $dx/dt = e^t$, so $dx = e^t dt$

$$\begin{aligned}\int \log(x)^2 dx &= \int \log(e^t)^2 e^t dt = \int t^2 e^t dt \\ &= (t^2 - 2t + 2)e^t = (\log(x))^2 - 2\log(x) + 2)x\end{aligned}$$

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- ▶ To find $\int f(x) dx$, put x equal to some function of t .
 - ▶ Find dx/dt , and rearrange to express dx in terms of t and dt .
 - ▶ Rewrite the integral in terms of t and dt .
 - ▶ Evaluate the integral, then rewrite the result in terms of x .

Examples I

- ▶ $\int \tan(x) dx = \int \frac{\sin(x)}{\cos(x)} dx = - \int \frac{\cos'(x)}{\cos(x)} dx = -\log(\cos(x)).$
- ▶ Consider $\int x^2 \tan(x^3) dx$. Put $u = x^3$, so $du = 3x^2 dx$, so $dx = du/(3x^2)$.

$$\begin{aligned}\int x^2 \tan(x^3) dx &= \int x^2 \tan(u) \frac{du}{3x^2} = \frac{1}{3} \int \tan(u) du = -\log(\cos(u))/3 \\ &= -\log(\cos(x^3))/3\end{aligned}$$

- ▶ Consider $\int x e^{\sqrt{x}} dx$. Put $t = \sqrt{x}$, so $x = t^2$, so $dx = 2t dt$.

$$\begin{aligned}\int x e^{\sqrt{x}} dx &= \int t^2 e^t \cdot 2t dt = 2 \int t^3 e^t dt = 2(t^3 - 3t^2 + 6t - 6)e^t \\ &= (2x^{3/2} - 6x + 12x^{1/2} - 12)e^{\sqrt{x}}\end{aligned}$$

Examples II

$$\begin{aligned}\int (2(x^2 + 1)e^x)^2 dx &= \int (4x^4 + 8x^2 + 4)e^{2x} dx \\ &= (Ax^4 + Bx^3 + Cx^2 + Dx + E)e^{2x} \\ (4x^4 + 8x^2 + 4)e^{2x} &= \frac{d}{dx}((Ax^4 + Bx^3 + Cx^2 + Dx + E)e^{2x}) \\ &= (4Ax^3 + 3Bx^2 + 2Cx + D)e^{2x} + \\ &\quad (Ax^4 + Bx^3 + Cx^2 + Dx + E) \cdot 2e^{2x} \\ &= e^{2x}(2Ax^4 + (4A + 2B)x^3 + (3B + 2C)x^2 + \\ &\quad (2C + 2D)x + (D + 2E))\end{aligned}$$

So $4 = 2A$, $0 = 4A + 2B$, $8 = 3B + 2C$, $0 = 2C + 2D$, $4 = D + 2E$
So $A = 2$, $B = -4$, $C = 10$, $D = -10$, $E = 7$

$$\int (2(x^2 + 1)e^x)^2 dx = (2x^4 - 4x^3 + 10x^2 - 10x + 7)e^{2x}.$$

Examples III

$$\begin{aligned} \int 1 + \cosh(x) + \cosh(x)^2 dx &= \int 1 + \frac{e^x + e^{-x}}{2} + \left(\frac{e^x + e^{-x}}{2}\right)^2 dx \\ &= \frac{1}{4} \int 4 + 2e^x + 2e^{-x} + e^{2x} + 2 + e^{-2x} dx \\ &= \frac{1}{4} \left(6x + 2e^x - 2e^{-x} + \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x} \right) \\ &= \frac{3}{2}x + \frac{e^x - e^{-x}}{2} + \frac{1}{4} \frac{e^{2x} - e^{-2x}}{2} \\ &= \frac{3}{2}x + \sinh(x) + \frac{1}{4} \sinh(2x). \end{aligned}$$

Examples IV

- To show that $\int \frac{dx}{\cos(x)} = \log\left(\frac{1 + \sin(x)}{\cos(x)}\right)$:

$$\begin{aligned}\frac{d}{dx} \left(\frac{1 + \sin(x)}{\cos(x)} \right) &= \frac{\cos(x) \cdot \cos(x) - (1 + \sin(x))(-\sin(x))}{\cos(x)^2} \\ &= \frac{\cos(x)^2 + \sin(x)^2 + \sin(x)}{\cos(x)^2} = \frac{1 + \sin(x)}{\cos(x)^2}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx} \log\left(\frac{1 + \sin(x)}{\cos(x)}\right) &= \left(\frac{1 + \sin(x)}{\cos(x)}\right)^{-1} \frac{d}{dx} \left(\frac{1 + \sin(x)}{\cos(x)}\right) \\ &= \frac{\cos(x)}{1 + \sin(x)} \frac{1 + \sin(x)}{\cos(x)^2} = \frac{1}{\cos(x)}\end{aligned}$$

Examples V

$$\begin{aligned} \int 8x \sin(x) \cos(x) dx &= \int 4x \sin(2x) dx \\ &= -2x \cos(2x) + \int 2 \cos(2x) dx \\ &= -2x \cos(2x) + \sin(2x). \end{aligned}$$

$$\text{Consider } \int 10e^{-x} \sin(x)^2 dx = \int 5e^{-x} dx + \int -5e^{-x} \cos(2x) dx.$$

$$\begin{aligned} \int -5e^{-x} \cos(2x) dx &= e^{-x}(A \cos(2x) + B \sin(2x)) \\ -5e^{-x} \cos(2x) &= e^{-x}((2B - A) \cos(2x) - (2A + B) \sin(2x)) \\ A &= 1, \quad B = -2 \end{aligned}$$

$$\int 10e^{-x} \sin(x)^2 dx = -5e^{-x} + e^{-x} \cos(2x) - 2e^{-x} \sin(2x).$$