Taylor Series

Introduction

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{k=0}^{\infty} kx^k \qquad (\text{ for } |x| < 1)$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

- For any reasonable function $f(x)$, there are coefficients a_k such that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

(when x is sufficiently small). This is the *Taylor series* for $f(x)$.

Exceptions

Not every function has a Taylor series.

- $f_0(x) = 1/x$ does not, because $f_0(0)$ is undefined.
- $f_1(x) = |x|$ and $f_2(x) = x^{1/3}$ do not, because the slopes $f_1'(0)$ and $f_2'(0)$ are not defined.
- $f_3(x) = \ln(x)$ does not, because $f_3'(x)$ is undefined for $x < 0$.
- $f_4(x) = e^{-1/x^2}$ does not, for a more subtle reason.

For a full explanation, see Level 3 complex analysis.

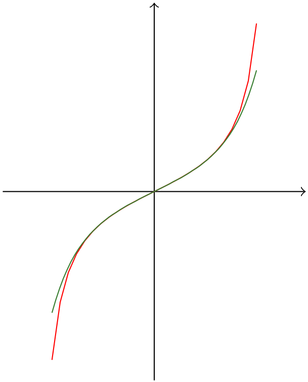
Truncated series

Often we only calculate with finitely many terms of the Taylor series.

$$\tan(x) = x + x^3/3 + 2x^5/15 + O(x^7)$$

The notation $O(x^7)$ means that there are extra terms involving powers x^k with $k \geq 7$. The above is the *7th order Taylor series* for $\tan(x)$. It is a good approximation to $\tan(x)$ if x is sufficiently small.

$\tan(x) = x + x^3/3 + 2x^5/15 + 17x^7/315 + O(x^9)$



Finding coefficients

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = \frac{1}{k!} \left. \frac{d^k y}{dx^k} \right|_{x=0}$$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \text{where} \quad a_k = f^{(k)}(0)/k!$$

Example:

$$\exp^{(k)}(x) = \cdots = \exp'''(x) = \exp''(x) = \exp'(x) = \exp(x) = e^x$$

$$\exp^{(k)}(0) = \cdots = \exp'''(0) = \exp''(0) = \exp'(0) = \exp(0) = 1$$

Thus $a_k = 1/k!$, and $\exp(x) = \sum_k x^k/k!$.

Another example

Take $f(x) = \sin(x)$.

$$
\begin{array}{llll}
f(x) = \sin(x) & f'(x) = \cos(x) & f''(x) = -\sin(x) & f'''(x) = -\cos(x) \\
f^{(4)}(x) = \sin(x) & f^{(5)}(x) = \cos(x) & f^{(6)}(x) = -\sin(x) & f^{(7)}(x) = -\cos(x) \\
f^{(8)}(x) = \sin(x) & f^{(9)}(x) = \cos(x) & f^{(10)}(x) = -\sin(x) & f^{(11)}(x) = -\cos(x)
\end{array}
$$

$$
\begin{array}{llll}
f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f'''(0) = -1 \\
f^{(4)}(0) = 0 & f^{(5)}(0) = 1 & f^{(6)}(0) = 0 & f^{(7)}(0) = -1 \\
f^{(8)}(0) = 0 & f^{(9)}(0) = 1 & f^{(10)}(0) = 0 & f^{(11)}(0) = -1
\end{array}
$$

$$
\begin{array}{llll}
a_0 = 0 & a_1 = 1 & a_2 = 0 & a_3 = -1/3! \\
a_4 = 0 & a_5 = 1/5! & a_6 = 0 & a_7 = -1/7! \\
a_8 = 0 & a_9 = 1/9! & a_{10} = 0 & a_{11} = -1/11!
\end{array}
$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Other methods

It is often easiest to deduce a Taylor series from known series for other functions.

$$
\begin{aligned}
e^{-x^2} &= \sum_k \frac{(-x^2)^k}{k!} = \sum_k (-1)^k \frac{x^{2k}}{k!} \\
\cosh(x) &= (e^x + e^{-x})/2 = \sum_k \frac{x^k + (-x)^k}{2\ (k!)} = \sum_{k\text{even}} \frac{x^k}{k!} = \sum_j \frac{x^{2j}}{(2j)!} \\
\sinh(x)/x &= (e^x - e^{-x})/(2x) = \sum_k \frac{x^k - (-x)^k}{2x\ (k!)} = \sum_{k\text{odd}} \frac{x^{k-1}}{k!} = \sum_j \frac{x^{2j}}{(2j+1)!} \\
1/(1-x) &= 1 + x + x^2 + x^3 + \ldots = \sum_k x^k \\
x \frac{d}{dx}\left(\frac{1}{1-x}\right) &= x \frac{d}{dx} \sum_k x^k = x \sum_k k\, x^{k-1} = \sum_k k\, x^k \\
x/(1-x)^2 &= \sum_k k\, x^k .
\end{aligned}
$$

Odd and even functions

Recall that $f(x)$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$. For example, $\cos(x)$ is even and $\sin(x)$ is odd. If

$$f(x) = \sum_k a_k x^k = \sum_{k \text{ even}} a_k x^k + \sum_{k \text{ odd}} a_k x^k$$

then

$$f(-x) = \sum_k a_k (-x)^k = \sum_{k \text{ even}} a_k x^k - \sum_{k \text{ odd}} a_k x^k.$$

Thus $f(x)$ is even iff the Taylor series involves only even powers of x, and $f(x)$ is odd iff the Taylor series involves only odd powers of x.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Algebra of series

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^7)$$

$$\begin{aligned}
\tan(x)^2 &= (x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5)^2 + O(x^7) \\
&= x^2 + \tfrac{1}{3}x^4 + \tfrac{2}{15}x^6 + \\
&\quad \tfrac{1}{3}x^4 + \tfrac{1}{9}x^6 + \tfrac{2}{45}x^8 \\
&\quad \tfrac{2}{15}x^6 + \tfrac{2}{45}x^8 + \tfrac{4}{225}x^{10} + O(x^7) \\
&= x^2 + \tfrac{2}{3}x^4 + \tfrac{17}{45}x^6 + O(x^7).
\end{aligned}$$

Expansion about other points

We can also expand $f(x)$ in terms of powers $(x-\alpha)^k$, for any α. More precisely,

$$f(x) = \sum_{k=0}^{\infty} b_k (x-\alpha)^k, \quad \text{where} \quad b_k = f^{(k)}(\alpha)/k!$$

	$\ln'(x) = x^{-1}$	$\ln''(x) = -x^{-2}$	$\ln'''(x) = 2x^{-3}$	$\ln^{(4)}(x) = -6x^{-4}$
$\ln(1) = 0$	$\ln'(1) = 1$	$\ln''(1) = -1$	$\ln'''(1) = 2$	$\ln^{(4)}(1) = -6$
$b_0 = 0$	$b_1 = 1$	$b_2 = -1/2$	$b_3 = 2/3! = 1/3$	$b_4 = -6/4! = -1/4$

$$\ln(x) = (x-1) - (x-1)^2/2 + (x-1)^3/3 - (x-1)^4/4 + O((x-1)^5).$$

More examples

We will find the series for $\tan(x)$ near $x = \frac{\pi}{4}$.

$$f(x) = \tan(x)$$
$$f'(x) = \frac{1}{\cos(x)^2}$$
$$f''(x) = -2\cos(x)^{-3}.-\sin(x) = \frac{2\sin(x)}{\cos(x)^3}$$

$$f(\tfrac{\pi}{4}) = 1 \qquad f'(\tfrac{\pi}{4}) = \frac{1}{(2^{-1/2})^2} = 2 \qquad f''(\tfrac{\pi}{4}) = \frac{2.2^{-1/2}}{(2^{-1/2})^3} = 4$$

$$a_0 = 1/0! = 1 \qquad a_1 = 2/1! = 2 \qquad a_2 = 4/2! = 2$$

$$\tan(x) = 1 + 2(x - \tfrac{\pi}{4}) + 2(x - \tfrac{\pi}{4})^2 + O((x - \tfrac{\pi}{4})^3).$$

More examples

Consider $y = x/(e^x - 1)$.

$$\begin{aligned}
e^x &= 1 + x + x^2/2 + x^3/6 + O(x^4) \\
e^x - 1 &= x + x^2/2 + x^3/6 + O(x^4) \\
\frac{1}{y} = \frac{e^x - 1}{x} &= 1 + x/2 + x^2/6 + O(x^3) = 1 + u + O(x^3) \qquad u = x/2 + x^2/6 \\
y = \frac{1}{1+u} &= 1 - u + u^2 + O(u^3) = 1 - u + u^2 + O(x^3) \\
u^2 &= x^2/4 + x^3/6 + x^4/36 = x^2/4 + O(x^3) \\
\frac{x}{e^x - 1} &= 1 - u + u^2 + O(x^3) \\
&= 1 - x/2 - x^2/6 + x^2/4 + O(x^3) = 1 - x/2 + x^2/12 + O(x^3)
\end{aligned}$$