

A warm-up

Exercises 1, 2 and 3 are covered by the online test for this week (available at <http://magic.shef.ac.uk/AiM>).

Exercise 1. Solve the equations

$$w + x + y + z = 1234 \quad (1) \qquad w + x - y - z = 1210 \quad (2)$$

$$w - x + y - z = 1010 \quad (3) \qquad w - x - y + z = 990 \quad (4)$$

Exercise 2. As you work through (a) to (g) below, draw a diagram showing all the curves, points and lines involved.

- Let C be the curve $x^2 + y^2 = 25$. Describe this geometrically. Note that the points $P = (-5, 0)$ and $Q = (3, 4)$ lie on C .
- Let L be the line joining P to Q . What is the equation for L ? What is its slope (or in other words, its gradient)?
- Let M be the line of slope -2 passing through Q . What is the equation for M ?
- Let R be the point (other than Q) where M meets C . What are the coordinates of R ?
- What is the angle between L and M ? (Note: here and almost everywhere else in University mathematics, angles are measured in radians, not degrees.) Do you know a general geometric fact that explains this?
- Let S be the midpoint of the line segment PQ . What are the coordinates of S ?
- Let N be the line joining S to the origin $O = (0, 0)$. Show that N is parallel to M .

Exercise 3. Differentiate the following functions, simplifying your answers as much as possible:

(a) $x + x^{10} + x^{100}$ (b) $(3x + 2)/(4x + 3)$ (c) $x \log(x) - x$ (d) $e^{-x} \sin(10x)$ (e) $\sin(x^2)$

Exercise 4. Evaluate the following integrals:

(a) $\int x^9 + x^{99} + x^{999} dx$ (b) $\int x e^{3x} dx$ (c) $\int x e^{-x^2} dx$ (d) $\int \frac{dx}{\sqrt{1-x^2}}$ (e) $\int_1^{e^2} \frac{dx}{x}$

Exercise 5. We have $10^3 = 1000 \approx 1024 = 2^{10}$. Deduce similar approximations for 10^9 and 8×10^9 as powers of 2.

- Exercise 6.** (a) Give a formula for the infinite sum $S = \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \dots$. (You may assume that $|x| < 1$.)
 (b) Deduce a formula for the infinite sum $T = \sum_{i=1}^{\infty} x^i = x + x^2 + x^3 + \dots$.
 (c) Consider the expression

$$y = 142857(10^{-6} + 10^{-12} + 10^{-18} + \dots).$$

Write the terms 142857×10^{-6} , 142857×10^{-12} and 142857×10^{-18} as decimals. What is the decimal expansion of y itself?

- (d) Using part (b), give an exact expression for $1/y$. (Your answer should be a whole number, not a fraction.)

Exercise 7. Consider the following functions:

$$\begin{array}{ll} \phi_1(x) = x - 1 & \phi_5(x) = x^4 + x^3 + x^2 + x + 1 \\ \phi_2(x) = x + 1 & \phi_6(x) = x^2 - x + 1 \\ \phi_3(x) = x^2 + x + 1 & \phi_{12}(x) = x^4 - x^2 + 1 \\ \phi_4(x) = x^2 + 1 & \end{array}$$

(These are called *cyclotomic polynomials*. They are very important in *Number Theory*, which means the study of prime numbers, divisibility and so on.)

Expand out the following products:

$$\phi_1(x)\phi_2(x) \quad \phi_1(x)\phi_3(x) \quad \phi_1(x)\phi_2(x)\phi_4(x) \quad \phi_1(x)\phi_5(x) \quad \phi_1(x)\phi_2(x)\phi_3(x)\phi_6(x).$$

Can you see the pattern? Can you guess what is the corresponding equation involving $\phi_{12}(x)$?

Exercise 8. Put

$$\begin{aligned} b_3(x) &= x(x-1)(x-2)/6 \\ b_4(x) &= x(x-1)(x-2)(x-3)/24 \\ b_5(x) &= x(x-1)(x-2)(x-3)(x-4)/120. \end{aligned}$$

- Simplify $b_4(x+1) - b_4(x)$. If you do this in the right way, it will take only a few simple steps; if you do it the wrong way, you will have to work much harder. Do not expand anything out if you do not have to.
- Simplify $b_5(x+1) - b_5(x)$.
- What is the general pattern? (Your answer should include a definition of $b_n(x)$ for all n .)