Maths with Maple — Week 1 tutorial – to be done **without Maple**

A warm-up

Exercises 1, 2 and 3 are covered by the online test for this week (available at `http://magic.shef.ac.uk/AiM`).

Exercise 1. Solve the equations

$$w + x + y + z = 1234 \quad (1) \qquad w + x - y - z = 1210 \quad (2)$$
$$w - x + y - z = 1010 \quad (3) \qquad w - x - y + z = 990 \quad (4)$$

Exercise 2. As you work through (a) to (g) below, draw a diagram showing all the curves, points and lines involved.

(a) Let C be the curve $x^2 + y^2 = 25$. Describe this geometrically. Note that the points $P = (-5, 0)$ and $Q = (3, 4)$ lie on C.
(b) Let L be the line joining P to Q. What is the equation for L? What is its slope (or in other words, its gradient)?
(c) Let M be the line of slope -2 passing through Q. What is the equation for M?
(d) Let R be the point (other than Q) where M meets C. What are the coordinates of R?
(e) What is the angle between L and M? (Note: here and almost everywhere else in University mathematics, angles are measured in radians, not degrees.) Do you know a general geometric fact that explains this?
(f) Let S be the midpoint of the line segment PQ. What are the coordinates of S?
(g) Let N be the line joining S to the origin $O = (0, 0)$. Show that N is parallel to M.

Exercise 3. Differentiate the following functions, simplifying your answers as much as possible:

(a) $x + x^{10} + x^{100}$ **(b)** $(3x + 2)/(4x + 3)$ **(c)** $x \log(x) - x$ **(d)** $e^{-x} \sin(10x)$ **(e)** $\sin(x^2)$

Exercise 4. Evaluate the following integrals:

(a) $\int x^9 + x^{99} + x^{999}\, dx$ **(b)** $\int x\, e^{3x}\, dx$ **(c)** $\int x\, e^{-x^2}\, dx$ **(d)** $\int \frac{dx}{\sqrt{1 - x^2}}$ **(e)** $\int_1^{e^2} \frac{dx}{x}$

Exercise 5. We have $10^3 = 1000 \approx 1024 = 2^{10}$. Deduce similar approximations for 10^9 and 8×10^9 as powers of 2.

Exercise 6. (a) Give a formula for the infinite sum $S = \sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \cdots$. (You may assume that $|x| < 1$.)
(b) Deduce a formula for the infinite sum $T = \sum_{i=1}^{\infty} x^i = x + x^2 + x^3 + \cdots$.
(c) Consider the expression

$$y = 142857(10^{-6} + 10^{-12} + 10^{-18} + \cdots).$$

Write the terms 142857×10^{-6}, 142857×10^{-12} and 142857×10^{-18} as decimals. What is the decimal expansion of y itself?
(d) Using part (b), give an exact expression for $1/y$. (Your answer should be a whole number, not a fraction.)

Exercise 7. Consider the following functions:

$$\begin{aligned} \phi_1(x) &= x - 1 & \qquad \phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\ \phi_2(x) &= x + 1 & \qquad \phi_6(x) &= x^2 - x + 1 \\ \phi_3(x) &= x^2 + x + 1 & \qquad \phi_{12}(x) &= x^4 - x^2 + 1 \\ \phi_4(x) &= x^2 + 1 \end{aligned}$$

(These are called *cyclotomic polynomials*. They are very important in *Number Theory*, which means the study of prime numbers, divisibility and so on.)

Expand out the following products:

$$\phi_1(x)\phi_2(x) \qquad \phi_1(x)\phi_3(x) \qquad \phi_1(x)\phi_2(x)\phi_4(x) \qquad \phi_1(x)\phi_5(x) \qquad \phi_1(x)\phi_2(x)\phi_3(x)\phi_6(x).$$

Can you see the pattern? Can you guess what is the corresponding equation involving $\phi_{12}(x)$?

Exercise 8. Put

$$b_3(x) = x(x-1)(x-2)/6$$
$$b_4(x) = x(x-1)(x-2)(x-3)/24$$
$$b_5(x) = x(x-1)(x-2)(x-3)(x-4)/120.$$

- Simplify $b_4(x+1) - b_4(x)$. If you do this in the right way, it will take only a few simple steps; if you do it the wrong way, you will have to work much harder. Do not expand anything out if you do not have to.
- Simplify $b_5(x+1) - b_5(x)$.
- What is the general pattern? (Your answer should include a definition of $b_n(x)$ for all n.)