Maths with Maple — Week 2 tutorial

Solving equations by hand

Exercises 1 and 3 appear on the online test for this week.

Exercise 1. Solve the equation

$$(((x^3+1)^3+1)^3+1)^3+1=9,$$

where x is a real number. (It is not helpful to expand out the left hand side.)

Exercise 2. Solve the equation $\ln(x)^3 = \ln(x)$.

Exercise 3. Find a solution to the equation $\cos(\theta)^2 + 2\cos(\theta) = 3$ where θ is a real number and $\theta > 10$.

Exercise 4. You are given that $x, y > 0$ and that

$$(A) \qquad x^3y^2 = e \qquad\qquad (B) \qquad x^4y^3 = e^2.$$

Find x and y. (One method is to take logs.)

Exercise 5. Solve the equation $\ln(e^{3t} - 7) = 0$.

Exercise 6. Solve the equation $\ln(e^x + 1) = x + 1$.

Exercise 7. Solve the following equations:

$$x = \sin(t) \qquad y = \cos(t) \qquad z = 6t/\pi \qquad x^2+y^2+z^2 = 10.$$

(In physics, this determines the time and place where an electron moving in a magnetic field hits the wall of a spherical chamber.)

Exercise 8. Given a constant $a > 1$, solve the following equations:

$$x^2 + y^2 = 1 \qquad\qquad (a-x)x - y^2 = 0.$$

If x and y satisfy the equations, what is $\sqrt{(a-x)^2+y^2}$?

(Let C be the circle of radius one centred at the origin, and let L be one of the two lines through $(a, 0)$ that just touches C; then L meets C at (x, y), where x and y satisfy the equations above.)

Exercise 9. Consider the equations

$$(1) \qquad a+2b+3c = 123 \qquad\qquad (2) \qquad 2a+3b+c = 231 \qquad\qquad (3) \qquad 3a+b+2c = 312.$$

Can you see a solution just by looking at the equations? Solve the equations by a more systematic method, and so verify that the visible solution is the only solution.

Exercise 10. Given a constant a, consider the following equations for x and y:

$$(A) \qquad x + ay = 1 \qquad\qquad (B) \qquad ax + y = a^2$$

(i) Solve the equations. Try to write your solution in a way that makes sense for $a = 1$.
(ii) What happens when $a = -1$? (You should go back to the original equations to answer this, rather than starting from your solution for the general case.)
(iii) What happens when $a = 1$? (First go back to the original equations, then compare with what you get by putting $a = 1$ in your solution to (i).)

Exercise 11. The hydrogen atoms in a molecule of methane lie at the points $(0, 0, 1)$, $(a, 0, -b)$, $(-c, d, -b)$ and $(-c, -d, -b)$, where $a, b, c, d > 0$ and the following equations are satisfied:

$$(1) \qquad c^2+d^2 = a^2 \qquad (2) \qquad c^2-d^2 = -ac \qquad (3) \qquad a^2+b^2 = 1 \qquad (4) \qquad b^2+b = ac.$$

Solve these equations to find a, b, c and d.

(**Hint:** find an equation involving only a and c, and put it in the form (something) $= 0$. Factorise it, and note that one of the factors is > 0, so the other one must be zero. This will let you write c in terms of a, and thus remove c from equation (4). You can then combine (3) and (4) to get an equation involving only b.)

Exercise 12. Solve the following equations for a, b, c and λ (assuming that all of these are positive):

$$bc = \lambda a \qquad ac = \lambda b \qquad ab = \lambda c \qquad a^2+b^2+c^2 = 1$$

Hence find the quantity $V = 8abc$. (This is the largest possible volume for a cuboid contained in a sphere of radius one. The maximising cuboid has sides of length $2a$, $2b$ and $2c$.)