Maths with Maple — Week 3 tutorial

Special functions 1

Exercise 1. Calculate the following:

$$\log_{10}(10000) \qquad \log_{100}(10000) \qquad \log_{1000}(10000) \qquad \log_{1/10}(\sqrt{1000})$$

Exercise 2. Some of the following statements are true, but others are either false or not meaningful. Decide which is which, and explain why. For each statement that is false, give an explicit counterexample. If there is a straightforward way to correct the statement, then do so.

(a) $\ln(e^x + e^y) = x + y$
(b) If $\ln(x) = a$, then $\ln(-x) = -a$
(c) $\log_a(b)\log_b(a) = 1$
(d) $\exp(\ln(x) - \ln(y)) = x - y$
(e) $\ln(\ln(\exp(\exp(x)))) = x$
(f) $\exp(\sqrt{x})^2 = \exp(x)$

Exercise 3. The following statements are false, but they can be corrected by changing some of the signs. Do so.

(a) $\sinh(x)^2 + \cosh(x)^2 = 1$
(b) $\cosh(2x) = \cosh(x)^2 - \sinh(x)^2$
(c) $\sinh(x + y) = -\sinh(x)\cosh(y) - \cosh(x)\sinh(y)$

Exercise 4. Simplify the following expressions (using the identities $\ln(xy) = \ln(x) + \ln(y)$, $\exp(x)^n = \exp(nx)$ and so on).

(a) $\ln(xy^2z^3)$
(b) $\ln(e^{(x^2)}e^{2xy}e^{(y^2)})$
(c) $\dfrac{\ln(a^n)\ln(a^m)}{\ln(a^p)\ln(a^q)}$
(d) $\dfrac{\exp(a + b\ln(t))}{\exp(a - b\ln(t))}$
(e) $\dfrac{\ln(a^n)\ln(b^m)}{\ln(a^m)\ln(b^n)}$

Exercise 5. Check the identities $\cosh(2x) = 2\cosh(x)^2 - 1$ and $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$.

Exercise 6. Simplify the expressions $4\cosh(x)^3 - 3\cosh(x)$ and $\sinh(x)^2\cosh(x)$.

Exercise 7. Using the fact that $\sin(x) = (e^{ix} - e^{-ix})/(2i)$ (where $i^2 = -1$), show that

$$4\sin(x)^3 = 3\sin(x) - \sin(3x)$$
$$4\sin(4x)\sin(2x)\sin(x) = -\sin(x) + \sin(3x) + \sin(5x) - \sin(7x).$$

Exercise 8. In this exercise, we show that $\operatorname{arcsinh}(y) = \ln(y + \sqrt{y^2 + 1})$.

(a) Suppose that $y = \sinh(x)$. Using the relation $\cosh(x)^2 - \sinh(x)^2 = 1$ and the definitions of sinh and cosh, simplify $y + \sqrt{y^2 + 1}$. Deduce that $x = \ln(y + \sqrt{y^2 + 1})$.
(b) Show in a similar way that if $z = \tanh(x)$, then $x = \ln((1 + z)/(1 - z))/2$.
(c) Now start instead with the formula $x = \ln(\sqrt{y^2 + 1} + y)$. Simplify $(\sqrt{y^2 + 1} + y)(\sqrt{y^2 + 1} - y)$, and rearrange to express $-x$ as ln(something else).
(d) Deduce that $\sinh(x) = y$.

Exercise 9. Write $s = \sin(\theta)$ and $c = \cos(\theta)$, for brevity, so that $s^2 + c^2 = 1$. Consider the expression

$$x = (2sc)^2 + (c^2 - s^2)^2.$$

Expand this out, factorise and simplify, and deduce that $x = 1$. What is the simple explanation for this?

Exercise 10. Write $f_n(x) = (1 + 2^{-n}x)^{2^n}$. Simplify $f_{n+1}(2x)f_n(x)^{-2}$.

Exercise 11. Consider the number $a = (\sqrt{6} + \sqrt{2})/4$.

(a) Simplify $2a^2 - 1$.
(b) What is $\cos(\pi/6)$?
(c) Give a formula relating $\cos(2\theta)$ to $\cos(\theta)$. What does this tell us about $\cos(\pi/6)$ and $\cos(\pi/12)$?
(d) Deduce that $\cos(\pi/12) = a$.