Maths with Maple — Week 4 tutorial

Special functions 2

Exercise 1. Find $\sin(-7\pi/3)$ and $\tan(9999\pi/4)$. (You should give exact answers, not decimal approximations.)

Exercise 2. If $t = \tan(\theta/2)$, show that $1 + t^2 = \cos(\theta/2)^{-2}$, and thus that

$$\sin(\theta) = \frac{2t}{1+t^2} \qquad \cos(\theta) = \frac{1-t^2}{1+t^2} \qquad \tan(\theta) = \frac{2t}{1-t^2}.$$

(It follows that any trigonometric function of θ can be rewritten as a rational function of t.)

Exercise 3. Simplify the expression $\sin(x)(\cos(x) + \cos(3x) + \cos(5x) + \cos(7x))$. Can you find a similar equation with five terms instead of four terms? What about six terms or seven terms or n terms?

Exercise 4. Show that $\tan(x + y) = (\tan(x) + \tan(y))/(1 - \tan(x)\tan(y))$. (It is easiest to do this by writing $\tan(z) = \sin(z)/\cos(z)$ and using the addition formulae for sin and cos, but you have to remember the signs correctly to make that work. You can also prove the formula be rewriting everything in terms of complex exponentials.)

Exercise 5. Show that $\cos(2\pi/5) = (\sqrt{5} - 1)/4$, as follows. Put $u = e^{2\pi i/5}$ and $c = \cos(2\pi/5) = (u + u^{-1})/2$. What is u^5? Expand out $(u - 1)u^2(4c^2 + 2c - 1)$, and deduce that $4c^2 + 2c - 1 = 0$. This gives two possibilities for c; explain why one of them can be rejected.

Exercise 6. Show that for $0 \leq \theta \leq \pi$ we have $\arcsin(\cos(\theta)) = \pi/2 - \theta$.

Exercise 7. (a) Simplify $(1 + \tan(\theta)^2)^{-1/2}$ and $\tan(\theta)(1 + \tan(\theta)^2)^{-1/2}$

(b) Deduce that $\cos(\arctan(t)) = (1 + t^2)^{-1/2}$ and $\sin(\arctan(t)) = t(1 + t^2)^{-1/2}$.

(c) Now suppose that $A, B > 0$ and $C = \sqrt{A^2 + B^2}$ and $\phi = \arctan(B/A)$. Simplify $C\cos(\phi)$ and $C\sin(\phi)$.

(d) Deduce that $A\sin(\theta) + B\cos(\theta) = C\sin(\theta + \phi)$.

Exercise 8. Let C be the curve given by equations $x = \cos(t)/(1 + e\cos(t))$ and $y = \sin(t)/(1 + e\cos(t))$. Show that $((1 - e^2)x + e)^2 + (1 - e^2)y^2 = 1$. (This actually implies that C is an ellipse; this calculation comes up in the theory of planetary motion. The number e is called the eccentricity, which measures how far the ellipse is from being a circle.)

Exercise 9. Show that $\sin(x)^4 + \cos(x)^4 = 1 - \frac{1}{2}\sin(2x)^2$ for all x.

Exercise 10. Let n be an integer bigger than one, and let R and r be numbers with $R > r > 0$. Expand and simplify the following equation, and thus find all the solutions:

$$(R\cos(x) + r\cos(nx))^2 + (R\sin(x) + r\sin(nx))^2 = (R + r)^2.$$

Exercise 11. Do not use Maple or a calculator for this problem, but instead analyse the situation logically. Sketch the graphs of the functions $\cos(x)^{100}$ and $\sin(x)^{100}$, thinking carefully about the maximum and minimum values and the overall shape. What exactly is the maximum value of $\cos(x)^{100}\sin(x)^{100}$? (You could use calculus for this, but it is not actually necessary.) How does this relate to your sketches?