

Special functions 2

Exercise 1. Find $\sin(-7\pi/3)$ and $\tan(9999\pi/4)$. (You should give exact answers, not decimal approximations.)

Exercise 2. If $t = \tan(\theta/2)$, show that $1 + t^2 = \cos(\theta/2)^{-2}$, and thus that

$$\sin(\theta) = \frac{2t}{1+t^2} \quad \cos(\theta) = \frac{1-t^2}{1+t^2} \quad \tan(\theta) = \frac{2t}{1-t^2}.$$

(It follows that any trigonometric function of θ can be rewritten as a rational function of t .)

Exercise 3. Simplify the expression $\sin(x)(\cos(x) + \cos(3x) + \cos(5x) + \cos(7x))$. Can you find a similar equation with five terms instead of four terms? What about six terms or seven terms or n terms?

Exercise 4. Show that $\tan(x+y) = (\tan(x) + \tan(y))/(1 - \tan(x)\tan(y))$. (It is easiest to do this by writing $\tan(z) = \sin(z)/\cos(z)$ and using the addition formulae for \sin and \cos , but you have to remember the signs correctly to make that work. You can also prove the formula by rewriting everything in terms of complex exponentials.)

Exercise 5. Show that $\cos(2\pi/5) = (\sqrt{5} - 1)/4$, as follows. Put $u = e^{2\pi i/5}$ and $c = \cos(2\pi/5) = (u + u^{-1})/2$. What is u^5 ? Expand out $(u - 1)u^2(4c^2 + 2c - 1)$, and deduce that $4c^2 + 2c - 1 = 0$. This gives two possibilities for c ; explain why one of them can be rejected.

Exercise 6. Show that for $0 \leq \theta \leq \pi$ we have $\arcsin(\cos(\theta)) = \pi/2 - \theta$.

- Exercise 7.**
- (a) Simplify $(1 + \tan(\theta)^2)^{-1/2}$ and $\tan(\theta)(1 + \tan(\theta)^2)^{-1/2}$
 - (b) Deduce that $\cos(\arctan(t)) = (1 + t^2)^{-1/2}$ and $\sin(\arctan(t)) = t(1 + t^2)^{-1/2}$.
 - (c) Now suppose that $A, B > 0$ and $C = \sqrt{A^2 + B^2}$ and $\phi = \arctan(B/A)$. Simplify $C \cos(\phi)$ and $C \sin(\phi)$.
 - (d) Deduce that $A \sin(\theta) + B \cos(\theta) = C \sin(\theta + \phi)$.

Exercise 8. Let C be the curve given by equations $x = \cos(t)/(1 + e \cos(t))$ and $y = \sin(t)/(1 + e \cos(t))$. Show that $((1 - e^2)x + e)^2 + (1 - e^2)y^2 = 1$. (This actually implies that C is an ellipse; this calculation comes up in the theory of planetary motion. The number e is called the eccentricity, which measures how far the ellipse is from being a circle.)

Exercise 9. Show that $\sin(x)^4 + \cos(x)^4 = 1 - \frac{1}{2} \sin(2x)^2$ for all x .

Exercise 10. Let n be an integer bigger than one, and let R and r be numbers with $R > r > 0$. Expand and simplify the following equation, and thus find all the solutions:

$$(R \cos(x) + r \cos(nx))^2 + (R \sin(x) + r \sin(nx))^2 = (R + r)^2.$$

Exercise 11. Do not use Maple or a calculator for this problem, but instead analyse the situation logically. Sketch the graphs of the functions $\cos(x)^{100}$ and $\sin(x)^{100}$, thinking carefully about the maximum and minimum values and the overall shape. What exactly is the maximum value of $\cos(x)^{100} \sin(x)^{100}$? (You could use calculus for this, but it is not actually necessary.) How does this relate to your sketches?