Maths with Maple — Week 6 tutorial

Differentiation 2

Exercises 13 and 14 will appear on the online test.

Exercise 1. Differentiate the following functions, simplifying your answers as much as possible:

(a) $x + x^{10} + x^{100}$ **(b)** $(3x+2)/(4x+3)$ **(c)** $x\ln(x) - x$ **(d)** $e^{-x}\sin(10x)$ **(e)** $\sin(x^2)$

Exercise 2. Find $\frac{d}{dx}\log(x + 2x^2 + 3x^3 + 4x^4)$.

Exercise 3. Find $\frac{d}{dx}\log(\cos(x))$.

Exercise 4. Let a, n and m be constants. Find $f'(x)$, where $f(x) = (x^n + a)^m$.

Exercise 5. Let a be a constant. Find $f'(x)$, where $f(x) = x^2\, e^{-1/(x+a)}$.

Exercise 6. Find $\frac{d}{dx}\cos\left(\left(\frac{x+1}{2}\right)^2\right)$.

Exercise 7. Let α, ω, a and b be constants, and put

$$f(t) = \sin((\omega + a\sin(\alpha t))t)$$
$$g(t) = (1 + b\sin(\alpha t))\sin(\omega t)$$

(These are FM and AM radio signals.) Find $f'(t)$ and $g'(t)$.

Exercise 8. Let a, b and ω be constants. Find $f'(x)$, where $f(x) = e^{-(x-a)^2/b}\sin(\omega x)$.

Exercise 9. If $y = \sqrt{2\pi}x^{x-1/2}e^{-x}$, show that $y'/y = \log(x) - 1/(2x)$.

Exercise 10. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions such that $f(1) = -1$ and $g(-1) = 1$. Show that $(f \circ g)'(-1) = (g \circ f)'(1)$. (**Hint:** You will need to use the chain rule, which can be written as $(p \circ q)'(x) = p'(q(x))\, q'(x)$.)

Exercise 11. Put $y = \exp(\exp(\exp(x)))$.

(a) Find dy/dx.
(b) Express x in terms of y.
(c) Working from (b), find dx/dy.
(d) Check that $\frac{dy}{dx}\frac{dx}{dy} = 1$.

Exercise 12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with $f(0) = 0$. Put $g(x) = f(f(f(f(x))))$. Use the chain rule to express $g'(x)$ in terms of derivatives of f, and thus show that $g'(0) \geq 0$.

Exercise 13. Let A, a, b and c be positive constants. What is the maximum value of the function $f(x) = A\exp(-ax^2 - bx - c)$?

Exercise 14. What is the maximum value of the function $f(x) = \sin(x) + \cos(x)$?

Exercise 15. Find dy/dx, where x and y are related as follows:

(a) $x^2 + xy + y^2 = 1$
(b) $e^x + e^y = x - y$
(c) $x\ln(x) + y\ln(y) = -1$

Exercise 16. Suppose that $x, y > 0$ and $x^{3/2} + y^{3/2} = 1$. Find dy/dx by implicit differentiation, and show that

$$\left(1 - \left(\frac{dy}{dx}\right)^3\right)^2 = \frac{1}{y^3}.$$

Exercise 17. In each of the following cases, find dy/dx. In two of the cases you should use $dy/dx = (dy/dt)/(dx/dt)$; in the other case, there is a much simpler way.

(a) $x = a\cos(nt)$, $y = b\sin(mt)$.
(b) $x = \tan(t)^{-1/2}$, $y = \tan(t)^{1/2}$.
(c) $x = e^t(\sin(t) + \cos(t))$, $y = e^t(\sin(t) - \cos(t))$.

Exercise 18. Suppose we have a thin rope wound around a pillar of radius one, and we hold the end taut and unwind it. After unwinding a length t of rope, it works out that the end of the rope is at position

$$(x, y) = (\cos(t) + t\sin(t), \sin(t) - t\cos(t)).$$

(a) Calculate dx/dt and dy/dt, and so find dy/dx in terms of t.
(b) Simplify $x^2 + y^2 - 1$, and so express dy/dx in terms of x and y.

Exercise 19. Show that if $y = \tan(x)$ then $y'''/(2y') = 2y^2 + y'$.

Exercise 20. Put $y = 1/(1-x)$. Calculate y', y'' and y'''. Guess a general formula for $y^{(n)}$

Exercise 21. Consider the function $y = Ae^x + Be^{2x} + Ce^{3x}$. Simplify the expressions

$$\tfrac{1}{2}y'' - \tfrac{5}{2}y' + 3y$$
$$-y'' + 4y' - 3y.$$

What should be the third part of the question?