

# Taylor series

**Exercise 1.** Find the following Taylor or MacLaurin series, to order 4 in each case.

- (a) The series for  $p(x) = 3e^x - 3e^{2x} + e^{3x}$  at  $x = 0$ .
- (b) The series for  $r(x) = e^x$  about  $x = 1$ .
- (c) The series for  $q(x) = \ln(1 - x)$  about  $x = 0$ .
- (d) The series for  $s(x) = 1 + x + x^2 + x^3 + x^4$  about  $x = 1$ .
- (e) The series for  $t(x) = 1/\cos(x)$  about  $x = 0$ . (Here you should remember that  $t(x)$  is an even function, and use this to simplify your calculation.)

**Exercise 2.** Put  $y = \tan(x)$  and recall that  $dy/dx = 1 + \tan(x)^2 = 1 + y^2$ . Differentiating this equation gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1 + y^2) = 2y \frac{dy}{dx} = 2y(1 + y^2) = 2y + 2y^3.$$

Continue this process to find  $d^k y/dx^k$  for  $k = 3, 4$  and  $5$ .

If we put  $x = 0$  then  $y = \tan(x) = \tan(0) = 0$  and so

$$\left. \frac{d^2y}{dx^2} \right|_{x=0} = 2 \times 0 + 2 \times 0^3 = 0.$$

In the same way, find the value of  $d^k y/dx^k$  for all  $k \leq 5$ , and so write down the 6th order Taylor series for  $\tan(x)$  at  $x = 0$ .

**Exercise 3.** Let  $f(x)$  be the 5th order Taylor series for  $e^x$  at  $x = 0$ . Write down  $f(x)$ . Expand out  $f(x)f(-x)$ , discarding any terms involving  $x^k$  for  $k \geq 5$ . What do you get, and why?

**Exercise 4.** Put  $y = 1/(1 - x)$ .

- (a) Find  $\frac{d^k y}{dx^k}$  for  $k = 1, 2, 3, 4$  and guess the general formula.
- (b) Hence write down the 4th order Taylor series for  $y$  at  $x = 0$ . We will call this  $u$ .
- (c) Expand out  $(1 - x)u$ , and explain the answer.
- (d) Work out the 4th order Taylor series for  $1/(1 - x)^2$ . Check that it is the same as what you get by squaring  $u$  and expanding it out, discarding terms of the form  $x^k$  with  $k \geq 4$ .

**Exercise 5.** Using the standard series

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

write down the 6th order Taylor series for  $\cos(x)$ ,  $\sin(x)$  and  $e^{ix}$  at  $x = 0$ . Check that to this order, the series for  $e^{ix}$  agrees with the series for  $\cos(x) + i \sin(x)$ . (Here  $i$  is the square root of  $-1$ , so  $i^2 = -1$ ,  $i^3 = -i$ ,  $i^4 = 1$  and so on.)

**Exercise 6.** The Bessel function  $y = J_2(x)$  has the form  $y = x^2/8 + ax^4 + O(x^6)$  for some constant  $a$ , and it satisfies the differential equation  $x^2 y'' + xy' + (x^2 - 4)y = 0$ . Starting with the given series for  $y$ , work out the series for  $x^2 y'' + xy' + (x^2 - 4)y$  (to order 6, again) and thus work out what the constant  $a$  must be.

**Exercise 7.** The 3rd order Taylor series for  $\sqrt{\cos(x)}$  at  $x = 0$  has the form  $\sqrt{\cos(x)} = 1 + ax + bx^2 + O(x^3)$  for some constants  $a$  and  $b$ . Square this and compare with the standard series for  $\cos(x)$ , and hence find  $a$  and  $b$ .

**Exercise 8.** Consider the function  $f(x) = (2x + 3)/(3x + 4)$ .

- (a) Calculate  $f'(x)$  and  $f''(x)$ .
- (b) Write down the third order Taylor series for  $f(x)$  at  $x = 0$ .
- (c) Observe that

$$f(x) = \frac{3 + 2x}{4} \frac{1}{1 - (-3x/4)}.$$

Using this and the standard geometric progression formula  $1/(1 - u) = \sum_{k=0}^{\infty} u^k$ , get another third order series for  $f(x)$ . Check that it is the same as in (b).

**Exercise 9.** Find numbers  $a$ ,  $b$  and  $c$  such that the function  $f(x) = a \frac{x-b}{x-c}$  has Taylor series  $8 + 2x + x^2 + O(x^3)$ .