Maths with Maple — Week 10 tutorial

Taylor series

Exercise 1. Find the following Taylor or MacLaurin series, to order 4 in each case.

(a) The series for $p(x) = 3e^x - 3e^{2x} + e^{3x}$ at $x = 0$.
(b) The series for $r(x) = e^x$ about $x = 1$.
(c) The series for $q(x) = \ln(1-x)$ about $x = 0$.
(d) The series for $s(x) = 1 + x + x^2 + x^3 + x^4$ about $x = 1$.
(e) The series for $t(x) = 1/\cos(x)$ about $x = 0$. (Here you should remember that $t(x)$ is an even function, and use this to simplify your calculation.)

Exercise 2. Put $y = \tan(x)$ and recall that $dy/dx = 1 + \tan(x)^2 = 1 + y^2$. Differentiating this equation gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1+y^2) = 2y\frac{dy}{dx} = 2y(1+y^2) = 2y + 2y^3.$$

Continue this process to find $d^k y/dx^k$ for $k = 3$, 4 and 5.

If we put $x = 0$ then $y = \tan(x) = \tan(0) = 0$ and so

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 \times 0 + 2 \times 0^3 = 0.$$

In the same way, find the value of $d^k y/dx^k$ for all $k \leq 5$, and so write down the 6th order Taylor series for $\tan(x)$ at $x = 0$.

Exercise 3. Let $f(x)$ be the 5th order Taylor series for e^x at $x = 0$. Write down $f(x)$. Expand out $f(x)f(-x)$, discarding any terms involving x^k for $k \geq 5$. What do you get, and why?

Exercise 4. Put $y = 1/(1-x)$.

(a) Find $\frac{d^k y}{dx^k}$ for $k = 1, 2, 3, 4$ and guess the general formula.
(b) Hence write down the 4th order Taylor series for y at $x = 0$. We will call this u.
(c) Expand out $(1-x)u$, and explain the answer.
(d) Work out the 4th order Taylor series for $1/(1-x)^2$. Check that it is the same as what you get by squaring u and expanding it out, discarding terms of the form x^k with $k \geq 4$.

Exercise 5. Using the standard series

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

write down the 6th order Taylor series for $\cos(x)$, $\sin(x)$ and e^{ix} at $x = 0$. Check that to this order, the series for e^{ix} agrees with the series for $\cos(x) + i\sin(x)$. (Here i is the square root of -1, so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and so on.)

Exercise 6. The Bessel function $y = J_2(x)$ has the form $y = x^2/8 + ax^4 + O(x^6)$ for some constant a, and it satisfies the differential equation $x^2y'' + xy' + (x^2-4)y = 0$. Starting with the given series for y, work out the series for $x^2y'' + xy' + (x^2-4)y$ (to order 6, again) and thus work out what the constant a must be.

Exercise 7. The 3rd order Taylor series for $\sqrt{\cos(x)}$ at $x = 0$ has the form $\sqrt{\cos(x)} = 1 + ax + bx^2 + O(x^3)$ for some constants a and b. Square this and compare with the standard series for $\cos(x)$, and hence find a and b.

Exercise 8. Consider the function $f(x) = (2x+3)/(3x+4)$.

(a) Calculate $f'(x)$ and $f''(x)$.
(b) Write down the third order Taylor series for $f(x)$ at $x = 0$.
(c) Observe that

$$f(x) = \frac{3+2x}{4}\,\frac{1}{1-(-3x/4)}.$$

Using this and the standard geometric progression formula $1/(1-u) = \sum_{k=0}^{\infty} u^k$, get another third order series for $f(x)$. Check that it is the same as in (b).

Exercise 9. Find numbers a, b and c such that the function $f(x) = a\frac{x-b}{x-c}$ has Taylor series $8 + 2x + x^2 + O(x^3)$.