MAS201 LINEAR MATHEMATICS FOR APPLICATIONS

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1. Introduction

From first year courses you will already be familiar with systems of linear equations, row-reduction and eigenvalue methods, with emphasis on dimensions two and three. This course will build on those ideas.

One theme that we will emphasise is the notion of an *algorithm*, or in other words a completely prescribed method that could be programmed into a computer that is guaranteed to solve a particular type of problem. It is an important conceptual point that almost all the problems in this course can be solved by a systematic set of algorithms, most of which involve setting up a matrix, row-reducing it, and reading off some kind of information from the result. These algorithms are similar to those used by programs such as Maple or the linear algebra functions in Python, and we will have various exercises using Python or Maple as well as exercises where we carry out algorithms by hand. Depending on your learning style you may wish to start by memorising the algorithms step by step. However, you should certainly aim to understand the conceptual background well enough that you can see why the algorithms work and can reconstruct them for yourself rather than learning them by rote.

We will also discuss some applications, including the following:

- Solution of certain systems of differential equations.
- Solution of difference equations.
- Markov chains as models of random processes.
- The Google PageRank algorithm for search engines.

2. Notation

Throughout these notes, the letters m and n will denote positive integers. Unless otherwise specified, the word *matrix* means a matrix where the entries are real numbers.

Throughout these notes, $\mathbb{R}$ will denote the set of all real numbers, and $\mathbb{Z}$ will denote the set of all integers. We will sometimes refer to a real number as a *scalar*.

We write $M_{m\times n}(\mathbb{R})$ to denote the set of all real $m \times n$ matrices, that is matrices with m rows and n columns, with real numbers as entries.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

a 2×3 matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

a 3×2 matrix

As a short-hand, we write $M_n(\mathbb{R})$ to stand for the set of real $n \times n$ (square) matrices. We write I_n for the $n \times n$ identity matrix, so $I_n \in M_n(\mathbb{R})$. For example

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By an *n-vector* we mean a column vector with n entries, which is the same as an $n \times 1$ matrix. We write $\mathbb{R}^n$ for the set of all n-vectors.

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by flipping A over, so the (i, j)'th entry in A^T is the same as the (j, i)'th entry in A. For example, we have

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}^T = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{bmatrix}.$$

Note also that the transpose of a row vector is a column vector, for example

$$\begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}^T = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}.$$

We will typically write column vectors in this way when it is convenient to lay things out horizontally.

We will write e_k for the k'th standard basis vector in $\mathbb{R}^n$, or equivalently the k'th column in the identity matrix I_n. (Here n is to be understood from the context.) For example, in the case $n = 4$ we have

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Maple syntax is as follows:

- A row vector such as $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ can be entered as <1|2|3>.
- A column vector such as $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can be entered as <1,2,3> (with commas instead of bars).

- A matrix such as $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ can be entered as `<<1|2|3>,<4|5|6>>`.
- To multiply a matrix by a (column) vector, or a matrix by another matrix, or to take the dot product of two vectors, use a dot. For example, if A has been set equal to a 2×3 matrix, and v has been set to a column vector of length 3, then you can type `A.v` to calculate the product Av.
- However, to multiply a vector or matrix by a scalar, you should use a star. If A and v are as above, you should type `6*A` and `7*v` to calculate $6A$ and $7v$.
- To calculate the transpose of A, you should type `Transpose(A)`. However, this will only work if you have previously loaded the linear algebra package, by typing `with(LinearAlgebra)`.
- The $n\times n$ identity matrix I_n can be entered as `IdentityMatrix(n)`. If you are working mostly with 3×3 matrices (for example) you may wish to enter $I3 := IdentityMatrix(3)$ as an abbreviation.

3. Products and transposes

We next recall some basic facts about products of matrices and transposes.

First, for column vectors $u, v \in \mathbb{R}^n$, we define the dot product by the usual rule

$$u.v = u_1v_1 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i.$$

For example, we have

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\cdot\begin{bmatrix}1000\\100\\10\\1\end{bmatrix} = 1000+200+30+4 = 1234.$$

Next, recall that we can multiply an $m\times n$ matrix by a vector in $\mathbb{R}^n$ to get a vector in $\mathbb{R}^m$.

Example 3.1.

$$\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}ax+by+cz\\ dx+ey+fz\end{bmatrix}$$

$$(2\times 3 \text{ matrix})(\text{vector in } \mathbb{R}^3) = (\text{vector in } \mathbb{R}^2)$$

One way to describe the general rule is as follows. Let A be an $m\times n$ matrix. We can divide A into n columns (each of which is a column vector in $\mathbb{R}^m$). If we write u_i for the i'th column, we get a decomposition

$$A = \left[\begin{array}{c|c|c} u_1 & \cdots & u_n \end{array}\right].$$

Alternatively, A has m rows, each of which is a row vector of length n, and so can be written as the transpose of a column vector in $\mathbb{R}^n$. If we write v_j for the transpose of the j'th row, we get a decomposition

$$A = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_m^T \end{array}\right].$$

Now let $t = \begin{bmatrix} t_1 & \cdots & t_n \end{bmatrix}^T$ be a vector in $\mathbb{R}^n$. The rule for multiplying a matrix by a vector is then

$$At = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_m^T \end{array}\right] t = \begin{bmatrix} v_1.t \\ \vdots \\ v_m.t \end{bmatrix}.$$

In Example 3.1 we have

$$v_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad v_2 = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \qquad t = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$At = \begin{bmatrix} v_1.t \\ v_2.t \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \end{bmatrix}$$

as expected.

On the other hand, it is not hard to see that the same rule can also be written in the form

$$At = t_1 u_1 + \cdots + t_n u_n.$$

In Example 3.1 we have

$$u_1 = \begin{bmatrix} a \\ d \end{bmatrix} \qquad u_2 = \begin{bmatrix} b \\ e \end{bmatrix} \qquad u_3 = \begin{bmatrix} c \\ f \end{bmatrix}$$

$$t_1 = x \qquad t_2 = y \qquad t_3 = z$$

so

$$t_1u_1 + t_2u_2 + t_3u_3 = x \begin{bmatrix} a \\ d \end{bmatrix} + y \begin{bmatrix} b \\ e \end{bmatrix} + z \begin{bmatrix} c \\ f \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \end{bmatrix} = At$$

as expected.

Example 3.2. Consider the case

$$A = \begin{bmatrix} 9 & 8 \\ 7 & 6 \\ 5 & 4 \\ 3 & 2 \end{bmatrix} \qquad t = \begin{bmatrix} 10 \\ 1000 \end{bmatrix} \qquad At = \begin{bmatrix} 8090 \\ 6070 \\ 4050 \\ 2030 \end{bmatrix}.$$

We have

$$A = \left[\begin{array}{c|c} u_1 & u_2 \end{array}\right] = \begin{bmatrix} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline v_4^T \end{bmatrix} \qquad t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 9 \\ 7 \\ 5 \\ 3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 8 \\ 6 \\ 4 \\ 2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 9 \\ 8 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$t_1 = 10$$

$$t_2 = 1000$$

The first approach gives

$$At = \begin{bmatrix} v_1.t \\ v_2.t \\ v_3.t \\ v_4.t \end{bmatrix} = \begin{bmatrix} 9 \times 10 + 8 \times 1000 \\ 7 \times 10 + 6 \times 1000 \\ 5 \times 10 + 4 \times 1000 \\ 3 \times 10 + 2 \times 1000 \end{bmatrix} = \begin{bmatrix} 8090 \\ 6070 \\ 4050 \\ 2030 \end{bmatrix},$$

and the second gives

$$At = t_1u_1 + t_2u_2 = 10\begin{bmatrix}9\\7\\5\\3\end{bmatrix} + 1000\begin{bmatrix}8\\6\\4\\2\end{bmatrix} = \begin{bmatrix}8090\\6070\\4050\\2030\end{bmatrix}.$$

As expected, the two answers are the same.

Next recall that the matrix product AB is only defined when the number of columns of A is the same as the number of rows of B. In other words, A must be an $m \times n$ matrix and B must be an $n \times p$ matrix for some positive integers n, m and p. It then works out that AB is an $n \times p$ matrix. To explain the rule for multiplication, we divide A into rows as before, and we divide B into columns, say

$$B = \left[\begin{array}{c|c|c} \\ w_1 & \cdots & w_p \\ \\ \end{array}\right]$$

Because A is an $m \times n$ matrix, we see that each of the vectors v_i has n entries. Because B is an $n \times p$ matrix, we see that each of the vectors w_j also has n entries. We can therefore form the dot product $v_i.w_j$. The product matrix AB is then given by

$$AB = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_m^T \end{array}\right] \left[\begin{array}{c|c|c} \\ w_1 & \cdots & w_p \\ \\ \end{array}\right] = \begin{bmatrix} v_1.w_1 & \cdots & v_1.w_p \\ \vdots & \ddots & \vdots \\ v_m.w_1 & \cdots & v_m.w_p \end{bmatrix}$$

Although you may not have seen it stated in precisely this way before, a little thought should convince you that this is just a paraphrase of the usual rule for multiplying matrices.

Remark 3.3. If A and B are numbers then of course $AB = BA$, but this does not work in general for matrices. Suppose that

A is an $m \times n$ matrix and B is an $n \times p$ matrix, so we can define AB as above.

(a) Firstly, BA may not even be defined. It is only defined if the number of columns of B is the same as the number of rows of A, or in other words $p = m$.

(b) Suppose that $p = m$, so A is an $m \times n$ matrix, and B is an $n \times m$ matrix, and both AB and BA are defined. We find that AB is an $m \times m$ matrix and BA is an $n \times n$ matrix. Thus, it is not meaningful to ask whether $AB = BA$ unless $m = n$.

(c) Suppose that $m = n = p$, so both A and B are square matrices of shape $n \times n$. This means that AB and BA are also $n \times n$ matrices. However, they are usually not equal. For example, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 10 & 10 & 10 \\ 100 & 100 & 100 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 20 & 20 & 20 \\ 300 & 300 & 300 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 10 & 10 & 10 \\ 100 & 100 & 100 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \\ 100 & 200 & 300 \end{bmatrix}.$$

Proposition 3.4. *If A is an $m \times n$ matrix and B is an $n \times p$ matrix then $(AB)^T = B^T A^T$.*

We first note that the dimensions match up so that this makes sense. As discussed above, the product AB is an $m \times p$ matrix, so $(AB)^T$ is a $p \times m$ matrix. On the other hand, B^T is a $p \times n$ matrix and A^T is an $n \times m$ matrix so $B^T A^T$ can be defined and it is another $p \times m$ matrix.

Note, however, that it would **not** generally be true (or even meaningful) to say that $(AB)^T = A^T B^T$: to make things work properly, the order of A and B **must** be reversed on the right hand side. Indeed, as A^T is an $n \times m$ matrix and B^T is a $p \times n$ matrix then $A^T B^T$ is not even defined unless $p = m$.

To prove the proposition, we decompose A into rows and B into columns as before. This gives

$$AB = \left[\begin{array}{c} u_1^T \\ \hline \vdots \\ \hline u_m^T \end{array}\right] \left[\begin{array}{c|c|c} v_1 & \cdots & v_p \end{array}\right] = \begin{bmatrix} u_1.v_1 & \cdots & u_1.v_p \\ \vdots & \ddots & \vdots \\ u_m.v_1 & \cdots & u_m.v_p \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} u_1.v_1 & \cdots & u_m.v_1 \\ \vdots & \ddots & \vdots \\ u_1.v_p & \cdots & u_m.v_p \end{bmatrix} = \begin{bmatrix} v_1.u_1 & \cdots & v_1.u_m \\ \vdots & \ddots & \vdots \\ v_p.u_1 & \cdots & v_p.u_m \end{bmatrix}$$

$$B^T A^T = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_p^T \end{array}\right] \left[\begin{array}{c|c|c} u_1 & \cdots & u_m \end{array}\right] = \begin{bmatrix} v_1.u_1 & \cdots & v_1.u_m \\ \vdots & \ddots & \vdots \\ v_p.u_1 & \cdots & v_p.u_m \end{bmatrix}$$

Example 3.5. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ we have

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}^T = \begin{bmatrix} ap+br & cp+dr \\ aq+bs & cq+ds \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} pa+rb & pc+rd \\ qa+sb & qc+sd \end{bmatrix} = (AB)^T.$$

4. Matrices and linear equations

We next recall the familiar process of conversion between matrix equations and systems of linear equations. For example, the system

$$
\begin{aligned}
w + 2x + 3y + 4z &= 1 \\
5w + 6x + 7y + 8z &= 10 \\
9w + 10x + 11y + 12z &= 100
\end{aligned}
$$

is equivalent to the single matrix equation

$$
\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 100 \end{bmatrix}.
$$

Similarly, the system

$$
\begin{aligned}
a + b + c &= 1 \\
a + 2b + 4c &= 2 \\
a + 3b + 9c &= 3 \\
a + 4b + 16c &= 4 \\
a + 5b + 25c &= 5
\end{aligned}
$$

is equivalent to the single matrix equation

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.
$$

The only point to watch here is that we need to move all constants to the right hand side, move all variables to the left hand side, write the variables in the same order in each equation, and fill any gaps with zeros. For example, the system

$$\begin{aligned} p + 7s &= q + 1 \\ 5r + 1 &= 7q - p \\ r + s &= p + q \end{aligned}$$

can be written more tidily as

$$\begin{array}{ccccl} p & -q & +0r & +7s & = 1 \\ p & -7q & +5r & +0s & = -1 \\ p & +q & -r & -s & = 0, \end{array}$$

and then we can just read off the entries of the corresponding matrix equation

$$\begin{bmatrix} 1 & -1 & 0 & 7 \\ 1 & -7 & 5 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Note that this kind of process only works well for *linear* equations, where every term is either a constant or a constant times a variable. If we want to find x and y, and our equations involve terms like x^2 or xy or e^x, then we will need a different approach. This course focusses on linear equations, but towards the end we will show how matrices can be used in a less direct way to solve certain multivariable quadratic equations.

A matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be expressed more compactly by just writing down the augmented matrix $[\mathbf{A}|\mathbf{b}]$, where $\mathbf{b}$ is added to $\mathbf{A}$ as an extra column at the right hand end. For example, the augmented matrices for the three systems discussed above are

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 8 & 10 \\ 9 & 10 & 11 & 12 & 100 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 9 & 3 \\ 1 & 4 & 16 & 4 \\ 1 & 5 & 25 & 5 \end{array}\right]$$

If we want to record the names of the variables we can add them as an extra row, giving matrices as follows:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 8 & 10 \\ 9 & 10 & 11 & 12 & 100 \\ \hline w & x & y & z & \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 9 & 3 \\ 1 & 4 & 16 & 4 \\ 1 & 5 & 25 & 5 \\ \hline a & b & c & \end{array}\right]$$

5. Reduced row-echelon form

Definition 5.1. Let A be a matrix of real numbers. Recall that A is said to be in *reduced row-echelon form* (RREF) if the following hold:

RREF0: Any rows of zeros come at the bottom of the matrix, after all the nonzero rows.
RREF1: In any nonzero row, the first nonzero entry is equal to one. These entries are called *pivots*.
RREF2: In any nonzero row, the pivot is further to the right than the pivots in all previous rows.
RREF3: If a column contains a pivot, then all other entries in that column are zero.

We will also say that a system of linear equations (in a specified list of variables) is in RREF if the corresponding augmented matrix is in RREF.

If RREF0, RREF1 and RREF2 are satisfied but not RREF3 then we say that A is in (unreduced) row-echelon form.

Example 5.2. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & \mathbf{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathbf{1} & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & \mathbf{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here A is not in RREF because the middle row is zero and the bottom row is not, so condition RREF0 is violated. The matrix B is also not in RREF because the first nonzero entry in the top row is 2 rather than 1, which violates RREF1. The matrix C is not in RREF because the pivot in the bottom row is to the left of the pivots in the previous rows, violating RREF2. The matrix D is not in RREF because the last column contains a pivot and also another nonzero entry, violating RREF3. On the other hand, the matrix

$$E = \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & \mathbf{1} & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF.

Example 5.3. The system of equations ($x - z = 1$ and $y = 2$) is in RREF because it has augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{array} \right]$$

which is in RREF. The system of equations ($x + y + z = 1$ and $y + z = 2$ and $z = 3$) is not in RREF because it has augmented matrix

$$B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

which is not in RREF.

As we will recall in the next section, any system of equations can be converted to an equivalent system that is in RREF. It is then easy to read off whether the new system has any solutions, and if so, to find them. The general method is as follows (but it may be clearer to just look at the examples given afterwards).

Method 5.4. Suppose we have a system of linear equations corresponding to a matrix that is in RREF. We can then solve it as follows.

(a) Any row of zeros can just be discarded, as it corresponds to an equation $0 = 0$ which is always true.
(b) If there is a pivot in the very last column (to the right of the bar) then the corresponding equation is $0 = 1$ which is always false, so the system has no solutions.
(c) Now suppose that there is no pivot to the right of the bar, but that every column to the left of the bar has a pivot. Because of RREF3, this means that the only nonzero entries in the whole matrix are the 1's in the pivot positions, so each equation directly gives the value of one of the variables and we have a unique solution.
(d) Suppose instead that there is no pivot to the right of the bar, but that only some of the columns to the left of the bar contain pivots. Each column to the left of the bar corresponds to one of the variables. Variables corresponding to columns with pivots are called *dependent variables*; the others are called *independent variables.* If we move all independent variables to the

right hand side, then each equation expresses one dependent variable in terms of the independent variables. The independent variables can take any values that we choose (so there will be infinitely many solutions). If we just want one solution rather than all possible solutions, the simplest thing is to set all the independent variables to be zero.

In the following examples, we will use the variables w, x, y and z.

Example 5.5. The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is an instance of case (b). It corresponds to the system

$$\begin{aligned} w + z &= 0 \\ x + y &= 0 \\ 0 &= 1 \\ 0 &= 0 \end{aligned}$$

which has no solution. (A solution would mean a system of numbers w, x, y and z for which *all four* equations are true. The third equation can never be true, so there is no solution. The fact that we can solve the first two equations (and that the fourth one is always true) is not relevant here.)

Of course no one would be foolish enough to write down this system of equations directly. The point is that we can start with a complicated system of equations and then apply row-reduction to simplify it. The row-reduction process may lead to the equation $0 = 1$, in which case we will conclude that the original system had no solution.

Example 5.6. The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & 13 \end{array}\right]$$

corresponds to the system of equations $w = 10$, $x = 11$, $y = 12$ and $z = 13$. In this case the equations are the solution; nothing needs to be done. This is an instance of case (c) in method 5.4.

Example 5.7. The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 10 \\ 0 & 0 & 1 & 4 & 20 \end{array}\right]$$

corresponds to the system of equations

$$\begin{aligned} w + 2x + 3z &= 10 \\ y + 4z &= 20. \end{aligned}$$

There are pivots in the first and third columns, so the corresponding variables w and y are dependent whereas x and z are independent. After moving the independent variables to the right hand side we get

$$\begin{aligned} w &= 10 - 2x - 3z \\ y &= 20 - 4z \end{aligned}$$

which expresses the dependent variables in terms of the independent ones. As x and z can take any values, we see that there are infinitely many solutions. This is an instance of case (d). Here it may be useful to write the solution in the form

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 - 2x - 3z \\ x \\ 20 - 4z \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 20 \\ 0 \end{bmatrix} + x \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

6. Row operations

Definition 6.1. Let A be a matrix. The following operations on A are called *elementary row operations*:

ERO1: Exchange two rows.
ERO2: Multiply a row by a nonzero constant.
ERO3: Add a multiple of one row to another row.

Theorem 6.2. *Let A be a matrix.*

(a) *By applying a sequence of row operations to A, one can obtain a matrix B that is in RREF.*
(b) *Although there are various different sequences that reduce A to RREF, they all give the same matrix B at the end of the process.*

In a moment we will recall the method used to reduce a matrix to RREF. It is not too hard to analyse the method carefully and check that it always works as advertised, which proves part (a) of the theorem. It is more difficult to prove (b) directly, and we will not do so here, although an essentially equivalent fact will appear in Proposition 20.6. With a more abstract point of view, as in MAS277, it becomes much easier. Nonetheless, you should appreciate that (b) is an important point.

Method 6.3. To reduce a matrix A to RREF, we do the following.

(a) If all rows are zero, then A is already in RREF, so we are done.
(b) Otherwise, we find a row that has a nonzero entry as far to the left as possible. Let this entry be u, in the k'th column of the j'th row say. Because we went as far to the left as possible, all entries in columns 1 to $k-1$ of the matrix are zero.

(c) We now exchange the first row with the j'th row (which does nothing if j happens to be equal to one).
(d) Next, we multiply the first row by u^{-1}. We now have a 1 in the k'th column of the first row.
(e) We now subtract multiples of the first row from all the other rows to ensure that the k'th column contains nothing except for the pivot in the first row.
(f) We now ignore the first row and apply row operations to the remaining rows to put them in RREF.
(g) If we put the first row back in, we have a matrix that is nearly in RREF, except that the first row may have nonzero entries above the pivots in the lower rows. This can easily be fixed by subtracting multiples of those lower rows.

While step (f) looks circular, it is not really a problem. Row-reducing a matrix with only one row is easy. If we start with two rows, then when we get to step (f) we need to row-reduce a matrix with only one row, which we can do; thus, the method works when there are two rows. If we start with three rows, then in step (f) we need to row-reduce a matrix with two rows, which we can do; thus, the method works when there are three rows. The pattern continues in the obvious way, which could be formalised as a proof by induction.

The method given above will work for any matrix, but in particular cases it may be possible to make the calculation quicker by performing row operations in a different order. By part (b) of Theorem 6.2, this will not affect the final answer.

Example 6.4. Consider the following sequence of reductions:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{2}$$

$$\begin{bmatrix}1&2&1&-1&2\\0&0&-2&-1&-13\\-1&-2&0&-1&-8\end{bmatrix}\xrightarrow{3}\begin{bmatrix}1&2&1&-1&2\\0&0&-2&-1&-13\\0&0&1&-2&-6\end{bmatrix}\xrightarrow{4}$$

$$\begin{bmatrix}1&2&1&-1&2\\0&0&1&1/2&13/2\\0&0&1&-2&-6\end{bmatrix}\xrightarrow{5}\begin{bmatrix}1&2&1&-1&2\\0&0&1&1/2&13/2\\0&0&0&-5/2&-25/2\end{bmatrix}\xrightarrow{6}$$

$$\begin{bmatrix}1&2&1&-1&2\\0&0&1&1/2&13/2\\0&0&0&1&5\end{bmatrix}\xrightarrow{7}\begin{bmatrix}1&2&1&-1&2\\0&0&1&0&4\\0&0&0&1&5\end{bmatrix}\xrightarrow{8}$$

$$\begin{bmatrix}1&2&0&-1&-2\\0&0&1&0&4\\0&0&0&1&5\end{bmatrix}\xrightarrow{9}\begin{bmatrix}1&2&0&0&3\\0&0&1&0&4\\0&0&0&1&5\end{bmatrix}$$

At stage 1 we exchange the first two rows as in step (c) of the method. At stage 2 we multiply the first row by -1 as in step (d), then at stage 3 we add the first row to the third row as in (e). As in step (f), we now ignore the first row temporarily and row-reduce the remaining two rows. There is nothing further to the left than the -2 on the second row, so we do not need to do any swapping. We divide the second row by -2 (stage 4) then subtract the second row from the third (stage 5). We are now back at step (f): we need to ignore the first two rows and row-reduce the last one. This just means multiplying by $-2/5$, which we do at stage 6. To complete the row-reduction of the bottom two rows, we just need to subtract half the bottom row from the middle row, which is stage 7. To complete the row-reduction of the whole matrix, we need to clear the entries in row 1 above the pivots in rows 2 and 3 as in step (g). We do this by subtracting the middle row from the top row (stage 8) and then adding the bottom row to the top row (stage 9).

Example 6.5. As another example, we will row-reduce the matrix

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix}.$$

The steps are as follows:

$$C \to \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & -5 & -5 & -5 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & -3 & -4 & -6 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Remark 6.6. We can ask Maple to do row reductions for us using the function `ReducedRowEchelonForm`. This will only work if we have already loaded the linear algebra package, and it is convenient to introduce a shorter name for the function at the same time. For example, we can check Example 6.4 as follows:

```
with(LinearAlgebra):
RREF := ReducedRowEchelonForm:

A :=
 << 0 |  0 | -2 | -1 | -13 >,
  <-1 | -2 | -1 |  1 |  -2 >,
  <-1 | -2 |  0 | -1 |  -8 >>;

RREF(A);
```

This will just give the final result of row-reduction, without any intermediate steps. If you want to check your working you can instead enter

```
with(Student[LinearAlgebra]):
GJET := GaussJordanEliminationTutor:

GJET(A);
```

This will open a new window in which you can click various buttons and so on to apply row operations. The system should be fairly self-explanatory.

The equivalent in Python with SymPy is to enter

```
A = Matrix(
   [[ 0 ,  0 , -2 , -1 , -13 ],
    [-1 , -2 , -1 ,  1 ,  -2 ],
    [-1 , -2 ,  0 , -1 ,  -8 ]]
)
A.rref()[0]
```

Remark 6.7. The following principle is sometimes useful. Suppose we have a matrix A, and that A can be converted to A' by some sequence of row operations. Suppose that B is obtained by deleting some columns from A, and that B' is obtained by deleting the corresponding columns from A'. When we perform row operations, the different columns do not interact in any way. It follows that B' can be obtained from B by performing the same sequence of row operations that we used to obtain A' from A.

For example, take

$$A = \begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

so Example 6.4 tells us that $A \to A'$. Now delete the middle column:

$$B = \begin{bmatrix} 0 & 0 & -1 & -13 \\ -1 & -2 & 1 & -2 \\ -1 & -2 & -1 & -8 \end{bmatrix} \qquad B' = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

The above principle tells us that B can be converted to B' by row operations. Note, however, that in this case B' is not in RREF; if we want an RREF matrix, we need to perform some additional row operations. In general B' may or may not be in RREF depending on which columns we delete.

Theorem 6.8. *Let A be an augmented matrix, and let A' be obtained from A by a sequence of row operations. Then the system of equations corresponding to A has the same solutions (if any) as the system of equations corresponding to A'.*

This should be fairly clear. The three types of elementary row operations correspond to reordering our system of equations, multiplying both sides of one equation by a nonzero constant, and adding one equation to another one. None of these operations changes the solution set. We thus have the following method:

Method 6.9. To solve a system of linear equations:

(a) Write down the corresponding augmented matrix.
(b) Row-reduce it by Method 6.3.
(c) Convert it back to a new system of equations, which (by Theorem 6.8) will have exactly the same solutions as the old ones.
(d) Read off the solutions by Method 5.4.

Example 6.10. We will try to solve the equations

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 1 \\ 4x & + & 2y & + & 3z & = & -1 \\ 6x & + & 3y & - & z & = & 11 \end{array}$$

The corresponding augmented matrix can be row-reduced as follows:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 2 & 3 & -1 \\ 6 & 3 & -1 & 11 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -4 & 8 \end{array}\right] \xrightarrow{2}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -4 \end{array}\right] \xrightarrow{3} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

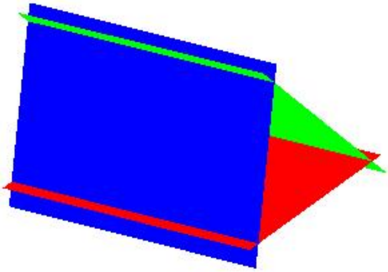
(At stage 1 we subtracted twice the first row from the second, and also subtracted three times the first row from the third. At stage 2 we added four times the second row to the third. At stage 3 we multiplied the last row by -1, then cleared the entries above all the pivots.)

The row-reduced matrix corresponds to the system

$$
\begin{aligned}
2x + y &= 0 \\
z &= 0 \\
0 &= 1,
\end{aligned}
$$

which has no solutions. This is an instance of case (b) in Method 5.4. We deduce that the original system of equations has no solutions either.

Geometrically, each of our three equations defines a plane in three-dimensional space, and by solving the three equations together we are looking for points where all three planes meet. Any two planes usually have a line where they intersect, and if we take the intersection with a third plane then we usually get a single point. However, this can go wrong if the planes are placed in a special way. In this example, the planes $2x+y+z = -1$ and $4x+2y+3z = -1$ intersect in the line where $z = -3$ and $y = 2-2x$. This runs parallel to the third plane where $6x+3y-z = 11$, but shifted sideways, so there is no point where all three planes meet.



Example 6.11. We will solve the equations

$$\begin{aligned} a+b+c+d &= 4 \\ a+b-c-d &= 0 \\ a-b+c-d &= 0 \\ a-b-c+d &= 0. \end{aligned}$$

The corresponding augmented matrix can be row-reduced as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & -2 & -2 & -4 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{4}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow{5} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{6}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{7} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Here, rather than slavishly following Method 6.3, we have applied row operations in a more creative order to make the structure of the equations clearer. The stages are as follows:

1. We subtract the first row from the second, and the third from the fourth.
2. We multiply the second and fourth rows by $-1/2$.

3. We subtract the second row from the first, and the fourth from the third.
4. We subtract the first row from the third, and the second from the fourth.
5. We multiply the third and fourth rows by $-1/2$.
6. We subtract the third row from the first, and the fourth from the second.
7. We exchange the second and third rows.

The final matrix corresponds to the equations $a = 1$, $b = 1$, $c = 1$ and $d = 1$, which give the unique solution to the original system of equations.

Remark 6.12. Often we want to solve a *homogeneous* equation $Ax = 0$, where the right hand side is zero. This means that the relevant augmented matrix is $[A|0]$. Row operations will not change the fact the last column is zero, so the RREF of $[A|0]$ will just be $[A'|0]$, where A' is the RREF of A. In this context we can save writing by leaving out the extra column and just working with A.

Example 6.13. Consider the homogeneous system

$$\begin{aligned} a + b + c + d + e + f &= 0 \\ 2a + 2b + 2c + 2d - e - f &= 0 \\ 3a + 3b - c - d - e - f &= 0 \end{aligned}$$

The corresponding unaugmented matrix can be row-reduced as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & -1 & -1 \\ 3 & 3 & -1 & -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(details are left to the reader). The final matrix corresponds to the homogeneous system

$$a + b = 0 \qquad c + d = 0 \qquad e + f = 0.$$

There are pivots in columns 1, 3 and 5, meaning that a, c and e are dependent variables, and b, d and f are independent. After moving the independent variables to the right hand side, the solution becomes $a = -b$, $c = -d$ and $e = -f$. If we prefer we can introduce new variables λ, μ and ν, and say that the general solution is

$$\begin{aligned} a &= -\lambda & c &= -\mu & e &= -\nu \\ b &= \lambda & d &= \mu & f &= \nu \end{aligned}$$

for arbitrary values of λ, μ and ν.

7. Linear combinations

Definition 7.1. Let $v_1, \dots, v_k$ and w be vectors in $\mathbb{R}^n$. We say that w is a *linear combination* of $v_1, \dots, v_k$ if there exist scalars $\lambda_1, \dots, \lambda_k$ such that

$$w = \lambda_1 v_1 + \cdots + \lambda_k v_k.$$

Example 7.2. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 10 \\ 100 \\ -111 \end{bmatrix}$$

If we take $\lambda_1 = 1$ and $\lambda_2 = 11$ and $\lambda_3 = 111$ we get

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 11 \\ -11 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 111 \\ -111 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 100 \\ -111 \end{bmatrix} = w,$$

which shows that w is a linear combination of v_1, v_2 and v_3.

Example 7.3. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

$$w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Any linear combination of $v_1, \dots, v_4$ has the form

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = \begin{bmatrix} 0 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ 2\lambda_1 + 4\lambda_2 + 8\lambda_3 + 16\lambda_4 \\ 3\lambda_1 + 9\lambda_2 + 27\lambda_3 + 81\lambda_4 \end{bmatrix}.$$

In particular, the first component of any such linear combination is zero. (You should be able to see this without needing to write out the whole formula.) As the first component of w is not zero, we see that w is *not* a linear combination of $v_1, \dots, v_4$.

Example 7.4. Consider the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

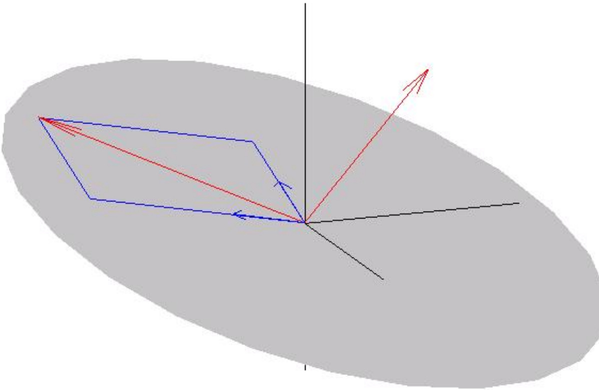
$$v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \qquad w = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Any linear combination of $v_1, \dots, v_5$ has the form

$$\lambda_1 v_1 + \cdots + \lambda_5 v_5 = \begin{bmatrix} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4 + 5\lambda_5 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \end{bmatrix}.$$

In particular, the first and last components of any such linear combination are the same. Again, you should be able to see this without writing the full formula. As the first and last components of w are different, we see that w is not a linear combination of $v_1, \dots, v_5$.

Example 7.5. Let v_1, v_2 and w be vectors in $\mathbb{R}^3$ (so we can think about them geometrically). For simplicity, assume that all three vectors are nonzero, and that v_1 and v_2 do not point in the same direction, nor do they point in opposite directions. This will mean that there is a unique plane P that passes through v_1, v_2 and the origin. It is not hard to see that P is just the set of all possible linear combinations of v_1 and v_2. Thus, our vector w is a linear combination of v_1 and v_2 if and only if w lies in the plane P.



We now want to explain a more systematic way to check whether a given vector is a linear combination of some given list of vectors. Note that for any k-vector $\lambda = \begin{bmatrix}\lambda_1 & \cdots & \lambda_k\end{bmatrix}^T$

we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_k \end{array}\right] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_k v_k,$$

which is the general form for a linear combination of $v_1, \ldots, v_k$. This makes it clear that w is a linear combination of $v_1, \ldots, v_k$ if and only if there is a vector λ which solves the matrix equation $A\lambda = w$. Using Theorem 6.8 we see that the equation $A\lambda = w$ has the same solutions as the equation $A'\lambda = w'$, which can be solved easily by Method 5.4. We thus arrive at the following method:

Method 7.6. Suppose we have vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ and another vector $w \in \mathbb{R}^n$, and we want to express w as a linear combination of the v_i (or show that this is not possible).

(a) We first let A be the matrix whose columns are the vectors v_i:

$$A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_k \end{array}\right] \in M_{n\times k}(\mathbb{R}).$$

(b) We then append w as an additional column to get an augmented matrix

$$B = \left[\begin{array}{c|c|c|c} v_1 & \cdots & v_k & w \end{array}\right] = \left[\begin{array}{c|c} A & w \end{array}\right].$$

This corresponds to the matrix equation $A\lambda = w$.

(c) Row-reduce B by Method 6.3 to get a matrix $B' = [A'|w']$ in RREF.

(d) If B' has a pivot in the last column, then w is not a linear combination of the vectors $v_1, \ldots, v_k$.

(e) If B' has no pivot in the last column, then we can use Method 5.4 to find a vector $\lambda = \begin{bmatrix} \lambda_1 & \cdots & \lambda_k \end{bmatrix}^T$ satisfying $A'\lambda = w'$. We then have $A\lambda = w$ and $\lambda_1 v_1 +$

$\cdots + \lambda_k v_k = w$, showing that w is a linear combination of $v_1, \dots, v_k$.

Example 7.7. Consider the vectors

$$v_1 = \begin{bmatrix} 11 \\ 11 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 11 \\ 11 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 11 \\ 11 \end{bmatrix} \qquad w = \begin{bmatrix} 121 \\ 221 \\ 1211 \\ 1111 \end{bmatrix}.$$

We ask whether w can be expressed as a linear combination $w = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$, and if so, what are the relevant values λ_1, λ_2 and λ_3? Following Method 7.6, we write down the augmented matrix $[v_1|v_2|v_3|w]$ and row-reduce it:

$$\left[\begin{array}{ccc|c} 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \\ 1 & 1 & 11 & 1111 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \end{array}\right] \xrightarrow{2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 0 & -10 & -120 & -12100 \\ 0 & 0 & -120 & -12000 \\ 0 & 10 & 0 & 100 \end{array}\right] \xrightarrow{3} \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 0 & 1 & 12 & 1210 \\ 0 & 0 & 1 & 100 \\ 0 & 1 & 0 & 10 \end{array}\right] \xrightarrow{4}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 11 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 100 \\ 0 & 1 & 0 & 10 \end{array}\right] \xrightarrow{5} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(1: move the bottom row to the top; 2: subtract multiples of row 1 from the other rows; 3: divide rows 2,3 and 4 by -10, -120 and 10; 4: subtract multiples of row 3 from the other rows; 5: subtract multiples of row 2 from the other rows.)

The final matrix corresponds to the system of equations

$$\lambda_1 = 1 \qquad \lambda_2 = 10 \qquad \lambda_3 = 100 \qquad 0 = 0$$

so we conclude that

$$w = v_1 + 10v_2 + 100v_3.$$

In particular, w can be expressed as a linear combination of v_1, v_2 and v_3. We can check the above equation directly:

$$v_1 + 10v_2 + 100v_3 = \begin{bmatrix} 11 \\ 11 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 10 \\ 110 \\ 110 \\ 10 \end{bmatrix} + \begin{bmatrix} 100 \\ 100 \\ 1100 \\ 1100 \end{bmatrix} = \begin{bmatrix} 121 \\ 221 \\ 1211 \\ 1111 \end{bmatrix} = w.$$

Example 7.8. Consider the vectors

$$a_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

To test whether b is a linear combination of a_1, a_2 and a_3, we write down the relevant augmented matrix and row-reduce it:

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ -1 & 0 & 3 & 2 \\ 0 & -1 & -2 & 3 \end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 3 & 6 & 5 \end{array}\right] \xrightarrow{3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 14 \end{array}\right] \xrightarrow{4}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{5} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(Stage 1: move the top row to the bottom, and multiply the other two rows by -1; Stage 2: subtract 2 times row 1 from row 3; Stage 3: subtract 3 times row 2 from row 3; Stage 4: divide row 3 by 14; Stage 5: subtract multiples of row 3 from rows 1 and 2.)

The last matrix has a pivot in the rightmost column, corresponding to the equation $0 = 1$. This means that the equation $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = b$ cannot be solved for λ_1, λ_2 and λ_3, or in other words that b is not a linear combination of a_1, a_2 and a_3.

We can also see this in a more direct but less systematic way, as follows. It is easy to check that $b.a_1 = b.a_2 = b.a_3 = 0$, which means that $b.(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) = 0$ for all possible choices of λ_1, λ_2 and λ_3. However, $b.b = 14 > 0$, so b cannot be equal to $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$.

8. Linear independence

Definition 8.1. Let $\mathcal{V} = v_1, \dots, v_k$ be a list of vectors in $\mathbb{R}^n$. A *linear relation* between the vectors v_i is a relation of the form $\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$, where $\lambda_1, \dots, \lambda_k$ are scalars. In other words, it is a way of expressing 0 as a linear combination of $\mathcal{V}$.

For any list we have the trivial linear relation $0v_1 + 0v_2 + \cdots + 0v_k = 0$. There may or may not be any nontrivial linear relations.

If the list $\mathcal{V}$ has a nontrivial linear relation, we say that it is a *linearly dependent* list. If the only linear relation on $\mathcal{V}$ is the trivial one, we instead say that $\mathcal{V}$ is *linearly independent*. We will often omit the word "linearly" for the sake of brevity.

Example 8.2. Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

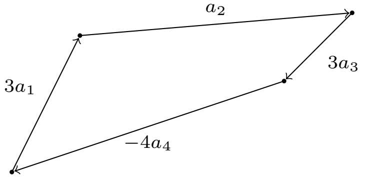
There is a nontrivial linear relation $v_1 + v_2 - v_3 - v_4 = 0$, so the list $\mathcal{V}$ is dependent.

Example 8.3. Consider the list $\mathcal{A}$ given by

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Just by writing it out, you can check that

$$3a_1 + a_2 + 3a_3 - 4a_4 = 0.$$



This is a nontrivial linear relation on the list $\mathcal{A}$, so $\mathcal{A}$ is dependent.

Example 8.4. Consider the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We claim that this is independent. To see this, consider a linear relation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$. Writing this out, we get

$$\begin{bmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 \\ \lambda_2 + \lambda_3 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By looking at the first and last rows we see that $\lambda_1 = \lambda_3 = 0$. By looking at the second row we get $\lambda_2 = -\lambda_1 = 0$ as well so our relation is the trivial relation. As the only linear relation is the trivial one, we see that $\mathcal{U}$ is independent.

Lemma 8.5. *Let v and w be vectors in $\mathbb{R}^n$, and suppose that $v \neq 0$ and that the list (v, w) is linearly dependent. Then there is a number α such that $w = \alpha v$.*

Proof. Because the list is dependent, there is a linear relation $\lambda v + \mu w = 0$ where λ and μ are not both zero. There are apparently three possibilities: (a) $\lambda \neq 0$ and $\mu \neq 0$; (b) $\lambda = 0$ and $\mu \neq 0$; (c) $\lambda \neq 0$ and $\mu = 0$. However, case (c) is not really possible. Indeed, in case (c) the equation $\lambda v + \mu w = 0$ would reduce to $\lambda v = 0$, and we could multiply by λ^{-1} to get $v = 0$; but $v \neq 0$ by assumption. In case (a) or (b) we can take $\alpha = -\lambda/\mu$ and we have $w = \alpha v$. $\square$

There is a systematic method using row-reduction for checking linear (in)dependence, as we will explain shortly. We first need a preparatory observation.

Definition 8.6. Let B be a $p \times q$ matrix. We say that B is *wide* if $p < q$, or *square* if $p = q$ or *tall* if $p > q$.

$$\underset{\text{wide}}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \qquad \underset{\text{square}}{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}} \qquad \underset{\text{tall}}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}$$

Lemma 8.7. *Let B be a $p \times q$ matrix in RREF.*

(a) *If B is wide then it is impossible for every column to contain a pivot.*
(b) *If B is square then the only way for every column to contain a pivot is if $B = I_q$.*
(c) *If B is tall then the only way for every column to contain a pivot is if B consists of I_q with $(p - q)$ rows of zeros added at the bottom (so $B = \left[\begin{array}{c} I_q \\ \hline 0_{(p-q)\times q} \end{array}\right]$).*

For example, the only 5×3 RREF matrix with a pivot in every column is this one:

$$\left[\begin{array}{c} I_3 \\ \hline 0_{2\times 3} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Proof. There is at most one pivot in every row, making at most p pivots altogether. If B is wide then we have q columns with $q > p$, so there are not enough pivots to have one in every column. This proves (a).

Now suppose instead that B does have a pivot in every column, so there are q pivots and we must have $p \geq q$. As B is in RREF we know that all entries above or below a pivot are zero. As there is a pivot in every column it follows that the pivots are the only nonzero entries in B. Every nonzero row contains precisely one pivot, so there must be q nonzero rows. The remaining $(p - q)$ rows are all zero, and they must occur at the bottom of B (because B is in RREF). Now the top $q \times q$ block contains q pivots which move to the right as we go down the matrix. It is easy to see that the only possibility for the top block is I_q, which proves (b) and (c). $\square$

Method 8.8. Let $\mathcal{V} = v_1, \ldots, v_m$ be a list of vectors in $\mathbb{R}^n$. We can check whether this list is dependent as follows.

(a) Form the $n \times m$ matrix

$$A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$$

whose columns are the vectors v_i.

(b) Row reduce A to get another $n \times m$ matrix B in RREF.

(c) If every column of B contains a pivot (so B has the form discussed in Lemma 8.7) then $\mathcal{V}$ is independent.
(d) If some column of B has no pivot, then the list $\mathcal{V}$ is dependent. Moreover, we can find the coefficients λ_i in a nontrivial linear relation by solving the vector equation $B\lambda = 0$ (which is easy because B is in RREF).

Remark 8.9. If $m > n$ then $\mathcal{V}$ is automatically dependent and we do not need to go through the method. (For example, any list of 5 vectors in $\mathbb{R}^3$ is automatically dependent, any list of 10 vectors in $\mathbb{R}^9$ is automatically dependent, and so on.) Indeed, in this case the matrices A and B are wide, so it is impossible for B to have a pivot in every column. However, this line of argument only tells us that there **exists** a nontrivial relation $\lambda_1 v_1 + \cdots + \lambda_m v_m = 0$, it does not tell us the coefficients λ_i. If we want to find the λ_i then we do need to go through the whole method as explained above.

We will give some examples of using the above method, and then explain why the method is correct.

Example 8.10. In example 8.2 we considered the list

$$v_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad v_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \qquad v_4 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix}1&0&1&0\\1&0&0&1\\0&1&0&1\\0&1&1&0\end{bmatrix} \xrightarrow{1} \begin{bmatrix}1&0&1&0\\0&0&-1&1\\0&1&0&1\\0&0&1&-1\end{bmatrix} \xrightarrow{2}$$

$$\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&-1\\0&0&-1&1\end{bmatrix} \xrightarrow{3} \begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

The end result has no pivot in the last column, so the original list is dependent. To find a specific linear relation, we solve the equation

$$\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

to get $\lambda_1 = -\lambda_4$, $\lambda_2 = -\lambda_4$ and $\lambda_3 = \lambda_4$ with λ_4 arbitrary. Taking $\lambda_4 = 1$ gives $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1, -1, 1, 1)$, corresponding to the relation $-v_1 - v_2 + v_3 + v_4 = 0$.

Example 8.11. In Example 8.3 we considered the list

$$a_1 = \begin{bmatrix}1\\2\end{bmatrix} \qquad a_2 = \begin{bmatrix}12\\1\end{bmatrix} \qquad a_3 = \begin{bmatrix}-1\\-1\end{bmatrix} \qquad a_4 = \begin{bmatrix}3\\1\end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent by Remark 8.9. Thus, there exist nontrivial linear relations

$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0.$$

To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix}1&12&-1&3\\2&1&-1&1\end{bmatrix} \to \begin{bmatrix}1&12&-1&3\\0&-23&1&-5\end{bmatrix} \to$$

$$\begin{bmatrix}1&12&-1&3\\0&1&-1/23&5/23\end{bmatrix} \to \begin{bmatrix}1&0&-11/23&9/23\\0&1&-1/23&5/23\end{bmatrix}$$

We now need to solve the matrix equation

$$\begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As this is in RREF, we can just read off the solution: $\lambda_1 = \frac{11}{23}\lambda_3 - \frac{9}{23}\lambda_4$ and $\lambda_2 = \frac{1}{23}\lambda_3 - \frac{5}{23}\lambda_4$ with λ_3 and λ_4 arbitrary. If we choose $\lambda_3 = 23$ and $\lambda_4 = 0$ we get $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (11, 1, 23, 0)$ so we have a relation

$$11a_1 + a_2 + 23a_3 + 0a_4 = 0.$$

(You should check directly that this is correct.) Alternatively, we can choose $\lambda_3 = 3$ and $\lambda_4 = -4$. Using the equations $\lambda_1 = \frac{11}{23}\lambda_3 - \frac{9}{23}\lambda_4$ and $\lambda_2 = \frac{1}{23}\lambda_3 - \frac{5}{23}\lambda_4$ we get $\lambda_1 = 3$ and $\lambda_2 = 1$ giving a different relation

$$3a_1 + a_2 + 3a_3 - 4a_4 = 0.$$

This is the relation that we observed in Example 8.3.

Example 8.12. In Example 8.4 we considered the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right]$$

The final matrix has a pivot in every column, as in Lemma 8.7. It follows that the list $\mathcal{U}$ is independent.

Proof of correctness of Method 8.8. Put

$$A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array} \right]$$

as in step (a) of the method, and let B be the RREF form of A. Note that for any vector $\lambda = \begin{bmatrix} \lambda_1 & \dots & \lambda_m \end{bmatrix}^T \in \mathbb{R}^m$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array} \right] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. By Theorem 6.8, these are the same as solutions to the equation $B\lambda = 0$, which can be found by Method 5.4. If there is a pivot in every column then none of the variables λ_i is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Thus, the only linear relation on $\mathcal{V}$ is the trivial one, which means that the list $\mathcal{V}$ is linearly independent.

Suppose instead that some column (the k'th one, say) does not contain a pivot. Then in Method 5.4 the variable λ_k will be independent, so we can choose $\lambda_k = 1$. This will give us a nonzero to solution to $B\lambda = 0$, or equivalently $A\lambda = 0$, corresponding to a nontrivial linear relation on $\mathcal{V}$. This shows that $\mathcal{V}$ is linearly dependent. □

9. Spanning sets

Definition 9.1. Suppose we have a list $\mathcal{V} = v_1, \dots, v_m$ of vectors in $\mathbb{R}^n$. We say that the list *spans* $\mathbb{R}^n$ if **every** vector in $\mathbb{R}^n$ can be expressed as a linear combination of $v_1, \dots, v_m$.

Example 9.2. Consider the list $\mathcal{V} = v_1, v_2, v_3, v_4$, where

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

In Example 7.3 we saw that the vector $w = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ is not a linear combination of this list, so the list $\mathcal{V}$ does not span $\mathbb{R}^4$.

Example 9.3. Consider the list $\mathcal{V} = v_1, v_2, v_3, v_4, v_5$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

In Example 7.4 we saw that the vector $w = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$ is not a linear combination of this list, so the list $\mathcal{V}$ does not span $\mathbb{R}^3$.

Example 9.4. Similarly, Example 7.8 shows that the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

does not span $\mathbb{R}^3$.

Example 9.5. Consider the list $\mathcal{U} = u_1, u_2, u_3$, where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

We will show that these span $\mathbb{R}^3$. Indeed, for any vector $v = \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3$ we can put

$$\lambda_1 = \frac{x+y-z}{2} \qquad \lambda_2 = \frac{x-y+z}{2} \qquad \lambda_3 = \frac{-x+y+z}{2}$$

and we find that

$$\begin{aligned}
&\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \\
&= \begin{bmatrix} (x+y-z)/2 \\ (x+y-z)/2 \\ 0 \end{bmatrix} + \begin{bmatrix} (x-y+z)/2 \\ 0 \\ (x-y+z)/2 \end{bmatrix} + \begin{bmatrix} 0 \\ (-x+y+z)/2 \\ (-x+y+z)/2 \end{bmatrix} \\
&= \begin{bmatrix} (x+y-z+x-y+z)/2 \\ (x+y-z-x+y+z)/2 \\ (x-y+z-x+y+z)/2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = v.
\end{aligned}$$

This expresses v as a linear combination of the list $\mathcal{U}$, as required.

Example 9.6. Consider the list $\mathcal{A} = a_1, a_2, a_3$ where

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^2$. Just by expanding out the right hand side, we see that

$$\begin{bmatrix} x \\ y \end{bmatrix} = (2y-4x) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (x-y) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

or in other words

$$v = (2y-4x)a_1 + (x-y)a_2 + xa_3.$$

This expresses an arbitrary vector $v \in \mathbb{R}^2$ as a linear combination of a_1, a_2 and a_3, proving that the list $\mathcal{A}$ spans $\mathbb{R}^2$.

In this case there are actually many different ways in which we can express v as a linear combination of a_1, a_2 and a_3. Another one is

$$v = (y-3x)a_1 + (2x-2y)a_2 + ya_3.$$

We now discuss a systematic method for spanning problems.

Method 9.7. Let $\mathcal{V} = v_1, \dots, v_m$ be a list of vectors in $\mathbb{R}^n$. We can check whether this list spans $\mathbb{R}^n$ as follows.

(a) Form the $m \times n$ matrix

$$C = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_m^T \end{array}\right]$$

whose rows are the row vectors v_i^T.

(b) Row reduce C to get another $m \times n$ matrix D in RREF.

(c) If every column of D contains a pivot (so D has the form discussed in Lemma 8.7) then $\mathcal{V}$ spans $\mathbb{R}^n$.

(d) If some column of D has no pivot, then the list $\mathcal{V}$ does not span $\mathbb{R}^n$.

Remark 9.8. This is almost exactly the same as Method 8.8, except that here we start by building a matrix whose rows are v_i^T, whereas in Method 8.8 we start by building a matrix whose columns are v_i. Equivalently, the matrix C in this method is the transpose of the matrix A in Method 8.8. Note, however, that transposing does not interact well with row-reduction, so the matrix D is **not** the transpose of B.

Remark 9.9. If $m < n$ then the matrices C and D above will be wide, so D cannot have a pivot in every column, so the list $\mathcal{V}$ cannot span $\mathbb{R}^n$. For example, no list of 4 vectors can span $\mathbb{R}^6$, and any list that spans $\mathbb{R}^8$ must contain at least 8 vectors and so on.

We will give some examples of using this method, then explain why it works.

Example 9.10. Consider the list

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

as in Example 9.2 (so $n = m = 4$). The relevant matrix C is

$$C = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \\ 0 & 1 & 16 & 81 \end{bmatrix}$$

The first column is zero, and will remain zero no matter what row operations we perform. Thus C cannot reduce to the identity matrix, so $\mathcal{V}$ does not span (as we already saw by a different method). In fact the row-reduction is

$$C \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

but it is not really necessary to go through the whole calculation.

Example 9.11. Consider the list

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

as in Example 9.3 (so $n = 3$ and $m = 5$). The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

At the end of the process the last column does not contain a pivot (so the top 3×3 block is not the identity), so the original list does not span. Again, we saw this earlier by a different method.

Example 9.12. For the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

in Example 9.4, the relevant row-reduction is

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 3 & -2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

In the last matrix the third column has no pivot, so the list does not span.

Example 9.13. Consider the list $\mathcal{U} = u_1, u_2, u_3$ from Example 9.5.

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\to\begin{bmatrix}1&1&0\\0&-1&1\\0&1&1\end{bmatrix}\to\begin{bmatrix}1&1&0\\0&1&-1\\0&0&2\end{bmatrix}\to$$

$$\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The end result is the identity matrix, so the list $\mathcal{U}$ spans $\mathbb{R}^3$.

Example 9.14. Consider the list $\mathcal{A}=\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix}$ from Example 9.6. The relevant row-reduction is

$$\begin{bmatrix}1&2\\2&3\\3&5\end{bmatrix}\to\begin{bmatrix}1&2\\0&-1\\0&-1\end{bmatrix}\to\begin{bmatrix}1&2\\0&1\\0&-1\end{bmatrix}\to\left[\begin{array}{cc}1&0\\0&1\\\hline 0&0\end{array}\right]$$

In the last matrix, the top 2×2 block is the identity. This means that the list $\mathcal{A}$ spans $\mathbb{R}^2$.

We now explain why Method 9.7 is valid.

Lemma 9.15. *Let C be an $m\times n$ matrix, and let C' be obtained from C by a single elementary row operation. Let s be a row vector of length n. Then s can be expressed as a linear combination of the rows of C if and only if it can be expressed as a linear combination of the rows of C'.*

Proof. Let the rows of C be $r_1,\dots,r_m$. Suppose that s is a linear combination of these rows, say

$$s=\lambda_1r_1+\lambda_2r_2+\lambda_3r_3+\cdots+\lambda_mr_m.$$

(a) Suppose that C' is obtained from C by swapping the first two rows, so the rows of C' are $r_2,r_1,r_3,\dots,r_m$. The sequence of numbers $\lambda_2,\lambda_1,\lambda_3,\dots,\lambda_m$ satisfies

$$s=\lambda_2r_2+\lambda_1r_1+\lambda_3r_3+\cdots+\lambda_mr_m,$$

which expresses s as a linear combination of the rows of C'. The argument is essentially the same if we exchange any other pair of rows.

(b) Suppose instead that C' is obtained from C by multiplying the first row by a nonzero scalar u, so the rows of C' are $ur_1, r_2, \ldots, r_m$. The sequence of numbers $u^{-1}\lambda_1, \lambda_2, \ldots, \lambda_m$ then satisfies

$$s = (u^{-1}\lambda_1)(ur_1) + \lambda_2 u_2 + \cdots + \lambda_m r_m,$$

which expresses s as a linear combination of the rows of C'. The argument is essentially the same if we multiply any other row by a nonzero scalar.

(c) Suppose instead that C' is obtained from C by adding u times the second row to the first row, so the rows of C' are $r_1+ur_2, r_2, r_3, \ldots, r_m$. The sequence of numbers $\lambda_1, \lambda_2 - u\lambda_1, \lambda_3, \ldots, \lambda_n$ then satisfies

$$\lambda_1(r_1 + ur_2) + (\lambda_2 - u\lambda_1)r_2 + \lambda_3 r_3 + \cdots + \lambda_m r_m =$$

$$\lambda_1 r_1 + \lambda_2 r_2 + \cdots + \lambda_m r_m = s,$$

which expresses s as a linear combination of the rows of C'. The argument is essentially the same if add a multiple of any row to any other row.

This proves half of the lemma: if s is a linear combination of the rows of C, then it is also a linear combination of the rows of C'. We also need to prove the converse: if s is a linear combination of the rows of C', then it is also a linear combination of the rows of C. We will only treat case (c), and leave the other two cases to the reader. The rows of C' are then $r_1 + ur_2, r_2, r_3, \ldots, r_m$. As s is a linear combination of these rows, we have $s = \mu_1(r_1 + ur_2) + \mu_2 r_2 + \cdots + \mu_m r_m$ for some numbers $\mu_1, \ldots, \mu_m$. Now the sequence of numbers

$\mu_1, (\mu_2 + u\mu_1), \mu_3, \ldots, \mu_m$ satisfies

$$s = \mu_1 r_1 + (\mu_2 + u\mu_1) r_2 + \mu_3 r_3 + \cdots + \mu_m r_m,$$

which expresses s as a linear combination of the rows of C. $\square$

Corollary 9.16. *Let C be an $m \times n$ matrix, and let D be obtained from C by a sequence of elementary row operation. Let s be a row vector of length n. Then s can be expressed as a linear combination of the rows of C if and only if it can be expressed as a linear combination of the rows of D.*

Proof. Just apply the lemma to each step in the row-reduction sequence. $\square$

Lemma 9.17. *Let D be an $m \times n$ matrix in RREF.*

(a) *Suppose that every column of D contains a pivot. Then $m \geq n$, the top $n \times n$ block of D is the identity, and everything below that block is zero. In this case every row vector of length n can be expressed as a linear combination of the rows of D.*
(b) *Suppose instead that the k'th column of D does not contain a pivot. Then the standard basis vector e_k* **cannot** *be expressed as a linear combination of the rows of D.*

Proof. (a) Suppose that every column of D contains a pivot. Lemma 8.7 tells us that $m \geq n$ and that $D = \left[\begin{array}{c} I_n \\ \hline 0_{(m-n)\times n} \end{array}\right]$. Thus, the first n rows are the standard basis vectors

$$
\begin{aligned}
r_1 &= e_1^T = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix} \\
r_2 &= e_2^T = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\end{bmatrix} \\
r_3 &= e_3^T = \begin{bmatrix}0 & 0 & 1 & \cdots & 0\end{bmatrix} \\
&\cdots\cdots\cdots\cdots \\
r_n &= e_n^T = \begin{bmatrix}0 & 0 & 0 & \cdots & 1\end{bmatrix}
\end{aligned}
$$

and $r_i = 0$ for $i > n$. This means that any row vector $v = \begin{bmatrix}v_1 & v_2 & \cdots & v_n\end{bmatrix}$ can be expressed as

$$
\begin{aligned}
v = &\begin{bmatrix}v_1 & 0 & 0 & \cdots & 0\end{bmatrix} + \\
&\begin{bmatrix}0 & v_2 & 0 & \cdots & 0\end{bmatrix} + \\
&\begin{bmatrix}0 & 0 & v_3 & \cdots & 0\end{bmatrix} + \\
&\cdots\cdots\cdots\cdots\cdots\cdots + \\
&\begin{bmatrix}0 & 0 & 0 & \cdots & v_n\end{bmatrix} \\
= &v_1r_1 + v_2r_2 + v_3r_3 + \cdots + v_nr_n,
\end{aligned}
$$

which is a linear combination of the rows of D.

(b) The argument here is most easily explained by an example. Consider the matrix

$$
D = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

This is in RREF, with pivots in columns 2, 5 and 8. Let r_i be the i'th row, and consider a linear combination

$$
s = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3
$$

$$
= \begin{bmatrix}0 & \lambda_1 & 2\lambda_1 & 3\lambda_1 & \lambda_2 & 4\lambda_1 + 6\lambda_2 & 5\lambda_1 + 7\lambda_2 & \lambda_3\end{bmatrix}.
$$

Note that the entries in the pivot columns 2, 5 and 8 of s are just the coefficients λ_1, λ_2 and λ_3. This is not a special feature of this example: it simply reflects the fact that pivot columns contain nothing except the pivot. Now choose a non-pivot column, say column number 6, and consider the standard basis vector e_6. Suppose we try to write e_6 as $\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3$, or in other words to solve

$$\begin{aligned} &[\ 0 \quad \mathbf{0} \quad 0 \quad 0 \quad \mathbf{0} \quad 1 \quad 0 \quad \mathbf{0}\ ] \\ =&[\ 0 \quad \boldsymbol{\lambda_1} \quad 2\lambda_1 \quad 3\lambda_1 \quad \boldsymbol{\lambda_2} \quad 4\lambda_1 + 6\lambda_2 \quad 5\lambda_1 + 7\lambda_2 \quad \boldsymbol{\lambda_3}\ ]. \end{aligned}$$

By looking in column 2, we see that λ_1 has to be zero. By looking in column 5, we see that λ_2 has to be zero. By looking in column 8, we see that λ_3 has to be zero. This means that $\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 = 0$, so $\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3$ cannot be equal to e_6.

This line of argument works more generally. Suppose that D is an RREF matrix and that the k'th column has no pivot. We claim that e_k is not a linear combination of the rows of D. We can remove any rows of zeros from D without affecting the question, so we may assume that every row is nonzero, so every row contains a pivot. Suppose that $e_k = \lambda_1 r_1 + \cdots + \lambda_m r_m$ say. By looking in the column that contains the first pivot, we see that $\lambda_1 = 0$. By looking in the column that contains the second pivot, we see that $\lambda_2 = 0$. Continuing in this way, we see that all the coefficients λ_i are zero, so $\sum_i \lambda_i r_i = 0$, which contradicts the assumption that $e_k = \lambda_1 r_1 + \cdots + \lambda_m r_m$.

$\square$

Proof of correctness of Method 9.7. We form a matrix C as in step (b) of the method. Recall that $\mathcal{V}$ spans $\mathbb{R}^n$ if and only if every column vector is a linear combination of the column vectors v_i. It is clear that this happens if and only if every row vector is a linear combination of the row vectors v_i^T, which are the rows of C. By Corollary 9.16, this happens if and only if every row vector is a linear combination of the rows of D. Lemma 9.17 tells us that this happens if and only if $m \geq n$ and D has a pivot in every column. $\square$

We can now prove the following result, which is one of a number of things that go by the name "duality".

Proposition 9.18. *Let P be an $m \times n$ matrix.*

(a) *The columns of P are linearly independent in $\mathbb{R}^m$ if and only if the columns of P^T span $\mathbb{R}^n$.*
(b) *The columns of P span $\mathbb{R}^m$ if and only if the columns of P^T are linearly independent in $\mathbb{R}^n$.*

Proof. Applying Method 8.8 to the columns of P is the same as applying Method 9.7 to the columns of P^T. Similarly, applying Method 9.7 to the columns of P is the same as applying Method 8.8 to the columns of P^T. $\square$

Remark 9.19. The way we have phrased the proposition reflects the fact that we have chosen to work with column vectors as far as possible. However, one can define what it means for row vectors to span or be linearly independent, in just the same way as we did for column vectors. We can then restate the proposition as follows:

(a) The columns of P are linearly independent if and only if the rows of P span.
(b) The columns of P span if and only if the rows of P are linearly independent.

10. Bases

Definition 10.1. A *basis* for $\mathbb{R}^n$ is a linearly independent list of vectors in $\mathbb{R}^n$ that also spans $\mathbb{R}^n$.

Remark 10.2. Any basis for $\mathbb{R}^n$ must contain precisely n vectors. Indeed, Remark 8.9 tells us that a linearly independent list can contain at most n vectors, and Remark 9.9 tells us that a spanning list must contain at least n vectors. As a basis has both these properties, it must contain precisely n vectors.

Example 10.3. Consider the list $\mathcal{U} = (u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For an arbitrary vector $v = \begin{bmatrix} a & b & c \end{bmatrix}^T$ we have

$$(a-b)u_1+(b-c)u_2+cu_3 = \begin{bmatrix} a-b \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b-c \\ b-c \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = v,$$

which expresses v as a linear combination of u_1, u_2 and u_3. This shows that $\mathcal{U}$ spans $\mathbb{R}^3$. Now suppose we have a linear relation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$. This means that

$$\begin{bmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_2 + \lambda_3 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

from which we read off that $\lambda_3 = 0$, then that $\lambda_2 = 0$, then that $\lambda_1 = 0$. This means that the only linear relation on $\mathcal{U}$ is the trivial one, so $\mathcal{U}$ is linearly independent. As it also spans, we conclude that $\mathcal{U}$ is a basis.

Proposition 10.4. *Suppose we have a list $\mathcal{V} = (v_1, \dots, v_n)$ of n vectors in $\mathbb{R}^n$, and we put*

$$A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array} \right]$$

(which is an $n \times n$ square matrix). Then $\mathcal{V}$ is a basis if and only if the equation $A\lambda = x$ has a **unique** *solution for every $x \in \mathbb{R}^n$.*

Proof. (a) Suppose that $\mathcal{V}$ is a basis. In particular, this means that an arbitrary vector $x \in \mathbb{R}^n$ can be expressed as a linear combination

$$x = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

Thus, if we form the vector $\lambda = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix}^T$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array} \right] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_n v_n = x,$$

so λ is a solution to the equation $A\lambda = x$. Suppose that μ is another solution, so we also have

$$\mu_1 v_1 + \cdots + \mu_n v_n = x.$$

By subtracting this from the earlier equation, we get

$$(\lambda_1 - \mu_1)v_1 + \cdots + (\lambda_n - \mu_n)v_n = 0.$$

This is a linear relation on the list $\mathcal{V}$. However, $\mathcal{V}$ is assumed to be a basis, so in particular it is linearly independent, so the only linear relation on $\mathcal{V}$ is the trivial one. This means that all the coefficients $\lambda_i - \mu_i$ are zero, so the vector λ is the same as the vector μ. In other words, λ is the **unique** solution to $A\lambda = x$, as required.

(b) We now need to prove the converse. Suppose that for every $x \in \mathbb{R}^n$, the equation $A\lambda = x$ has a unique solution. Equivalently, for every $x \in \mathbb{R}^n$, there is a unique sequence of coefficients $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 v_1 + \ldots + \lambda_n v_n = x$. Firstly, we can temporarily ignore the uniqueness, and just note that every element $x \in \mathbb{R}^n$ can be expressed as a linear combination of $v_1, \ldots, v_n$. This means that the list $\mathcal{V}$ spans $\mathbb{R}^n$. Next, consider the case $x = 0$. The equation $A\lambda = 0$ has $\lambda = 0$ as one solution. By assumption, the equation $A\lambda = 0$ has a unique solution, so $\lambda = 0$ is the only solution. Using the above equation for $A\lambda$, we can restate this as follows: the only sequence $(\lambda_1, \ldots, \lambda_n)$ for which $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ is the sequence $(0, \ldots, 0)$. In other words, the only linear relation on $\mathcal{V}$ is the trivial one. This means that $\mathcal{V}$ is linearly independent, and it also spans $\mathbb{R}^n$, so it is a basis.

$\square$

This gives us a straightforward method to check whether a list is a basis.

Method 10.5. Let $\mathcal{V} = (v_1, \ldots, v_m)$ be a list of vectors in $\mathbb{R}^n$.

(a) If $m \neq n$ then $\mathcal{V}$ is not a basis.

(b) If $m = n$ then we form the matrix

$$A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$$

and row-reduce it to get a matrix B.

(c) If $B = I_n$ then $\mathcal{V}$ is a basis; otherwise, it is not.

Proof of correctness of Method 10.5. Step (a) is justified by Remark 10.2, so for the rest of the proof we can assume that $n = m$.

Suppose that A row-reduces to I_n. Fix a vector $x \in \mathbb{R}^n$, and consider the equation $A\lambda = x$. This corresponds to the augmented matrix $[A|x]$. If we perform the same row operations on $[A|x]$ as we did to convert A to I_n, we will obtain a matrix of the form $[I_n|x']$. Theorem 6.8 tells us that the solutions to $A\lambda = x$ are the same as the solutions to $I_n\lambda = x'$, so it is clear that $\lambda = x'$ is the unique solution. Thus the hypothesis of Proposition 10.4 is satisfied, and we can conclude that $\mathcal{V}$ is a basis.

Now suppose instead that row-reduction of A leads to a matrix B in RREF that is not equal to I_n. We know that I_n is the only square RREF matrix with a pivot in every column, so B cannot have a pivot in every column. Method 8.8 therefore tells us that the list $\mathcal{V}$ is linearly dependent, so it cannot be a basis. $\square$

Example 10.6. Consider the vectors

$$v_1 = \begin{bmatrix}1\\2\\3\\2\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}3\\2\\1\\2\\3\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \qquad v_4 = \begin{bmatrix}1\\3\\5\\3\\1\end{bmatrix}$$

To decide whether they form a basis, we construct the corresponding matrix A and start row-reducing it:

$$\begin{bmatrix}1&3&1&1&5\\2&2&1&3&3\\3&1&1&5&1\\2&2&1&3&3\\1&3&1&1&5\end{bmatrix} \to \begin{bmatrix}1&3&1&1&5\\0&-4&-1&1&-7\\0&-8&-2&2&-14\\0&-4&-1&1&-7\\0&0&0&0&0\end{bmatrix} \to$$

$$\begin{bmatrix}1&3&1&1&5\\0&-4&-1&1&-7\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

Already after the first step we have a row of zeros, and it is clear that we will still have a row of zeros after we complete the row-reduction, so A does not reduce to the identity matrix, so the vectors v_i do not form a basis.

Example 10.7. Consider the vectors

$$p_1 = \begin{bmatrix} 1 \\ 1 \\ 11 \\ 1 \end{bmatrix} \qquad p_2 = \begin{bmatrix} 1 \\ 11 \\ 1 \\ 11 \end{bmatrix} \qquad p_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 11 \end{bmatrix} \qquad p_4 = \begin{bmatrix} 1 \\ 11 \\ 11 \\ 11 \end{bmatrix}$$

To decide whether they form a basis, we construct the corresponding matrix A and row reduce it:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 11 & 1 & 11 \\ 11 & 1 & 1 & 11 \\ 1 & 11 & 11 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 10 & 0 & 10 \\ 0 & -10 & -10 & 0 \\ 0 & 10 & 10 & 10 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After a few more steps, we obtain the identity matrix. It follows that the list p_1, p_2, p_3, p_4 is a basis.

Now suppose that the list $\mathcal{V} = v_1, \ldots, v_n$ is a basis for $\mathbb{R}^n$, and that w is another vector in $\mathbb{R}^n$. By the very definition of a basis, it must be possible to express w (in a unique way) as a linear combination $w = \lambda_1 v_1 + \cdots + \lambda_n v_n$. If we want to find the coefficients λ_i, we can use Method 7.6. That method can be streamlined slightly in this context, as follows.

Method 10.8. Let $\mathcal{V} = v_1, \dots, v_n$ be a basis for $\mathbb{R}^n$, and let w be another vector in $\mathbb{R}^n$.

(a) Let B be the matrix

$$B = \left[\begin{array}{c|c|c|c} v_1 & \cdots & v_n & w \end{array}\right] \in M_{n\times(n+1)}(\mathbb{R}).$$

(b) Let B' be the RREF form of B. Then B' will have the form $[I_n|\lambda]$ for some column vector

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

(c) Now $w = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

It is clear from our discussion of Method 7.6 and Method 10.5 that this is valid.

Example 10.9. We will express the vector $q = \begin{bmatrix} 0.9 \\ 0.9 \\ 0 \\ 10.9 \end{bmatrix}$ in terms of the basis p_1, p_2, p_3, p_4 introduced in Example 10.7. We form the relevant augmented matrix, and apply the same row-reduction steps as in Example 10.7, except that we now have an extra column.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 1 & 11 & 1 & 11 & 0.9 \\ 11 & 1 & 1 & 11 & 0 \\ 1 & 11 & 11 & 11 & 10.9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 10 & 0 & 10 & 0 \\ 0 & -10 & -10 & 0 & -9.9 \\ 0 & 10 & 10 & 10 & 10 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0.99 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0.99 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0.99 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0.9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0.99 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0.9 \\ 0 & 1 & 0 & 0 & -0.01 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -0.1 \\ 0 & 1 & 0 & 0 & -0.01 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right]$$

The final result is $[I_4|\lambda]$, where $\lambda = \begin{bmatrix} -0.1 & -0.01 & 1 & 0.01 \end{bmatrix}^T$. This means that q can be expressed in terms of the vectors p_i as follows:

$$q = -0.1p_1 - 0.01p_2 + p_3 + 0.01p_4.$$

Example 10.10. One can check that the vectors u_1, u_2, u_3 and u_4 below form a basis for $\mathbb{R}^4$.

$$u_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \qquad u_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \end{bmatrix} \qquad u_3 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{bmatrix} \qquad u_4 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \\ \frac{1}{7} \end{bmatrix}$$

We would like to express v in terms of this basis. The matrix formed by the vectors u_i is called the *Hilbert matrix*; it is notoriously hard to row-reduce. We will therefore use Maple:

```
with(LinearAlgebra):
RREF := ReducedRowEchelonForm;
u[1] := <1,1/2,1/3,1/4>;
u[2] := <1/2,1/3,1/4,1/5>;
```

```
u[3] := <1/3,1/4,1/5,1/6>;
u[4] := <1/4,1/5,1/6,1/7>;
v    := <1,1,1,1>;
B    := <u[1]|u[2]|u[3]|u[4]|v>;
RREF(B);
```

Maple tells us that

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/3 & 1/4 & 1 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & -180 \\ 0 & 0 & 0 & 1 & 140 \end{array}\right].$$

We conclude that

$$v = -4u_1 + 60u_2 - 180u_3 + 140u_4.$$

The equivalent in Python with SymPy is to enter

```
B = Matrix([
 [1,1/2,1/3,1/4],
 [1/2,1/3,1/4,1/5],
 [1/3,1/4,1/5,1/6],
 [1/4,1/5,1/6,1/7],
 [1,1,1,1]
]).T
B.rref()[0]
```

Proposition 10.11. *Let A be an $n \times n$ matrix. Then the columns of A form a basis for $\mathbb{R}^n$ if and only if the columns of A^T form a basis for $\mathbb{R}^n$.*

Proof. Suppose that the columns of A form a basis. This means in particular that the columns of A are linearly independent, so the columns of A^T span $\mathbb{R}^n$ by part (a) of Proposition 9.18. Also, the columns of A must span $\mathbb{R}^n$ (by the other half of the definition of a basis) so the columns of A^T

are linearly independent by part (b) of Proposition 9.18. As the columns of A^T are linearly independent and span $\mathbb{R}^n$, they form a basis.

The converse is proved in the same way. $\square$

Proposition 10.12. *Let $\mathcal{V}$ be a list of n vectors in $\mathbb{R}^n$ (so the number of vectors is the same as the number of entries in each vector).*

(a) *If the list is linearly independent then it also spans, and so is a basis.*
(b) *If the list spans then it is also linearly independent, and so is a basis.*

(However, these rules are **not valid** *for lists of length different from n.)*

Proof. Let A be the matrix whose columns are the vectors in $\mathcal{V}$.

(a) Suppose that $\mathcal{V}$ is linearly independent. Let B be the matrix obtained by row-reducing A. Method 8.8 tells us that B has a pivot in every column. As B is also square, we must have $B = I_n$. Method 10.5 therefore tells us that $\mathcal{V}$ is a basis.
(b) Suppose instead that $\mathcal{V}$ (which is the list of columns of A) spans $\mathbb{R}^n$. By Proposition 9.18, we conclude that the columns of A^T are linearly independent. Now A^T has n columns, so we can apply part (a) to deduce that the columns of A^T form a basis. By Proposition 10.11, the columns of A must form a basis as well.

$\square$

11. Elementary matrices and invertibility

Definition 11.1. Fix an integer $n > 0$. We define $n \times n$ matrices as follows.

(a) Suppose that $1 \leq p \leq n$ and that λ is a nonzero real number. We then let $D_p(\lambda)$ be the matrix that is the same as I_n except that $(D_p(\lambda))_{pp} = \lambda$.

(b) Supose that $1 \leq p, q \leq n$ with $p \neq q$, and that μ is an arbitrary real number. We then let $E_{pq}(\mu)$ be the matrix that is the same as I_n except that $(E_{pq}(\lambda))_{pq} = \mu$.

(c) Supose again that $1 \leq p, q \leq n$ with $p \neq q$. We let F_{pq} be the matrix that is the same as I_n except that $(F_{pq})_{pp} = (F_{pq})_{qq} = 0$ and $(F_{pq})_{pq} = (F_{pq})_{qp} = 1$.

An *elementary matrix* is a matrix of one of these types.

Example 11.2. In the case $n = 4$, we have

$$D_2(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{24}(\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Elementary matrices correspond precisely to row operations, as explained in the next result.

Proposition 11.3. *Let A be an $n \times n$ matrix, and let A' be obtained from A by a single row operation. Then $A' = UA$ for some elementary matrix U. In more detail:*

(a) *Let A' be obtained from A by multiplying the p'th row by λ. Then $A' = D_p(\lambda)A$.*

(b) *Let A' be obtained from A by adding μ times the q'th row to the p'th row. Then $A' = E_{pq}(\mu)A$.*

(c) *Let A' be obtained from A by exchanging the p'th row and the q'th row. Then $A' = F_{pq}A$.*

We will not give a formal proof, as examples are more illuminating: if we take

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

then

$$D_2(\lambda)A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} & \lambda a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$E_{24}(\mu)A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} + \mu a_{41} & a_{22} + \mu a_{42} & a_{23} + \mu a_{43} & a_{24} + \mu a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$F_{24}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

Corollary 11.4. *Let A and B be $n \times n$ matrices, and suppose that A can be converted to B by a sequence of row operations. Then $B = UA$ for some matrix U that can be expressed as a product of elementary matrices.*

Proof. The assumption is that there is a sequence of matrices $A_0, A_1, \ldots, A_r$ starting with $A_0 = A$ and ending with $A_r = B$ such that A_i is obtained from A_{i-1} by a single row operation. By the Proposition, this means that there is an elementary matrix U_i such that $A_i = U_i A_{i-1}$. This gives

$$\begin{aligned}
A_1 &= U_1 A_0 = U_1 A \\
A_2 &= U_2 A_1 = U_2 U_1 A \\
A_3 &= U_3 A_2 = U_3 U_2 U_1 A
\end{aligned}$$

and so on. Eventually we get $B = A_r = U_r U_{r-1} \cdots U_1 A$. We can thus take $U = U_r U_{r-1} \cdots U_1$ and we have $B = UA$ as required. $\square$

Theorem 11.5. *Let A be an $n \times n$ matrix. Then the following statements are equivalent: if any one of them is true then they are all true, and if any one of them is false then they are all false.*

(a) *A can be row-reduced to I_n.*
(b) *The columns of A are linearly independent.*
(c) *The columns of A span $\mathbb{R}^n$.*
(d) *The columns of A form a basis for $\mathbb{R}^n$.*
(e) *A^T can be row-reduced to I_n.*
(f) *The columns of A^T are linearly independent.*
(g) *The columns of A^T span $\mathbb{R}^n$.*
(h) *The columns of A^T form a basis for $\mathbb{R}^n$.*
(i) *There is a matrix U such that $UA = I_n$.*
(j) *There is a matrix V such that $AV = I_n$.*

Moreover, if these statements are all true then there is a unique matrix U that satisfies $UA = I_n$, and this is also the unique matrix that satisfies $AU = I_n$ (so the matrix V in (j) is necessarily the same as the matrix U in (i)).

Proof. It is clear from Propositions 10.11 and 10.12 that statements (a) to (h) are all equivalent. If these statements hold then in particular A row-reduces to I_n, so Corollary 11.4 tells us that there exists a matrix U with $UA = I_n$, so (i) holds. Similarly, if (a) to (h) hold then A^T row-reduces to the identity, so Corollary 11.4 tells us that there exists a matrix W with $WA^T = I_n$. Taking the transpose (and remembering Proposition 3.4) gives $AW^T = I_n$, so we can take $V = W^T$ to see that (j) holds.

Conversely, suppose that (i) holds. Let $v_1, \dots, v_r$ be the columns of A. As we have discussed previously, a linear relation $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ gives a a vector λ with $A\lambda = 0$. As $UA = I_n$ this gives $\lambda = UA\lambda = U0 = 0$, so our linear relation is the trivial one. We conclude that the columns v_i are linearly independent, so (b) holds (as do the equivalent statements (a) to (h)). Similarly, if we assume that (j) holds then $V^T A^T = I_n$ and we can use this to show that the columns of A^T are linearly independent, which is statement (f). It is now clear that all the statements (a) to (j) are equivalent.

Now suppose we have matrices U and V as in (i) and (j). Consider the product UAV. Using $UA = I_n$, we see that $UAV = V$. Using $AV = I_n$, we also see that $UAV = U$. It follows that $U = V$.

Moreover, this calculation also gives uniqueness. Suppose we have two matrices U_1 and U_2 with $U_1 A = U_2 A = I_n$. This means that (i) holds and (j) is equivalent to (i) so there is a matrix V with $AV = I_n$. BY considering $U_1 AV$ as before we see that $U_1 = V$. If we instead consider $U_2 AV$ we see that $U_2 = V$, so $U_1 = U_2$. Thus, there is a unique matrix satisfying (i). By a very similar argument, there is a unique matrix satisfying (j). $\square$

Definition 11.6. We say that A is *invertible* if (any one of) the conditions (a) to (j) in Theorem 11.5 hold. If so, we write A^{-1} for the unique matrix satisfying $A^{-1}A = I_n = AA^{-1}$ (which exists by the Theorem).

Remark 11.7. It is clear from Theorem 11.5 that A is invertible if and only if A^T is invertible.

Example 11.8. All elementary matrices are invertible. More precisely:

(a) $D_p(\lambda^{-1})D_p(\lambda) = I_n$, so $D_p(\lambda)$ is invertible with inverse $D_p(\lambda^{-1})$. For example, when $n = 4$ and $p = 2$ we have

$$D_2(\lambda)D_2(\lambda^{-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4.$$

(b) $E_{pq}(\mu)E_{pq}(-\mu) = I_n$, so $E_{pq}(\mu)$ is invertible with inverse $E_{pq}(-\mu)$. For example, when $n = 4$ and $p = 2$ and $q = 4$ we have

$$E_{24}(\mu)E_{24}(-\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4.$$

(c) $F_{pq}^2 = I_n$, so F_{pq} is invertible and is its own inverse. For example, when $n = 4$ and $p = 2$ and $q = 4$ we have

$$F_{24}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4.$$

Proposition 11.9. *If A and B are invertible $n \times n$ matrices, then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. More generally, if $A_1, A_2, \ldots, A_r$ are invertible $n \times n$ matrices, then the product $A_1 A_2 \cdots A_r$ is also invertible, with*

$$(A_1 A_2 \cdots A_r)^{-1} = A_r^{-1} \cdots A_2^{-1} A_1^{-1}.$$

Proof. For the first claim, put $C = AB$ and $D = B^{-1}A^{-1}$. We then have $DC = B^{-1}A^{-1}AB$, but the term $A^{-1}A$ in the middle is just I_n, so $DC = B^{-1}I_nB$. Here I_nB is just the same as B, so $DC = B^{-1}B = I_n$. Similarly $CD = ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$. This shows that D is an inverse for C, so C is invertible with $C^{-1} = D$ as claimed.

More the more general statement, we put $P = A_1 A_2 \cdots A_r$ and $Q = A_r^{-1} \cdots A_2^{-1} A_1^{-1}$. We then have

$$\begin{aligned} PQ &= A_1 \cdots A_{r-2} A_{r-1} A_r A_r^{-1} A_{r-1}^{-1} A_{r-2}^{-1} \cdots A_1^{-1} \\ &= A_1 \cdots A_{r-2} A_{r-1} A_{r-1}^{-1} A_{r-2}^{-1} \cdots A_1^{-1} \\ &= A_1 \cdots A_{r-2} A_{r-2}^{-1} \cdots A_1^{-1} \\ &= \cdots \\ &= A_1 A_1^{-1} \\ &= I_n. \end{aligned}$$

In the same way, we also see that $QP = I_n$. This shows that Q is an inverse for P. □

Corollary 11.10. *Let A and B be $n \times n$ matrices, and suppose that A can be converted to B by a sequence of row operations. Then $B = UA$ for some invertible matrix U.*

Proof. Corollary 11.4 tells us that $B = UA$ for some matrix U that is a product of elementary matrices. Example 11.8 and Proposition 11.9 then tell us that U is invertible. □

If we want to know whether a given matrix A is invertible, we can just row-reduce it and check whether we get the identity. We can find the inverse by a closely related procedure.

Method 11.11. Let A be an $n \times n$ matrix. Form the augmented matrix $[A|I_n]$ and row-reduce it. If the result has the form $[I_n|B]$, then A is invertible with $A^{-1} = B$. If the result has any other form then A is not invertible.

Proof of correctness. Let $[T|B]$ be the row-reduction of $[A|I_n]$. It follows that T is the row-reduction of A, so A is invertible if and only if $T = I_n$. Suppose that this holds, so $[A|I_n]$ reduces to $[I_n|B]$. As in Corollary 11.4 we see that there is a matrix U (which can be written as a product of elementary matrices) such that $[I_n|B] = U[A|I_n] = [UA|U]$. This gives $B = U$ and $UA = I_n$ so $BA = I_n$, so $B = A^{-1}$. □

Example 11.12. Consider the matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. We have the following row-reduction:

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

We conclude that $A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$. It is a straightforward exercise to check this directly:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Example 11.13. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$. We have the following row-reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right]$$

We conclude that

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}.$$

12. Determinants

We now record some important facts about determinants. The proofs are given in Appendix B, which will not be covered in lectures and will not be examinable. For completeness we start with the official definition:

Definition 12.1. Let A be an $n \times n$ matrix, and let a_{ij} denote the entry in the i'th row of the j'th column. We define

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where the sum runs over all permutations σ of the set $\{1, \ldots, n\}$, and $\operatorname{sgn}(\sigma)$ is the signature of σ.

Remark 12.2. In Python with SymPy you can calculate the determinant of A by entering `det(A)`. In Maple you need to enter `Determinant(A)`. As with many other matrix functions, this will only work if you have already entered `with(LinearAlgebra)` to load the linear algebra package. You may also wish to enter `det:=Determinant`, after which you will be able to use the shorter notation `det(A)`.

Further details are given in the appendix. In this section we will not use the definition directly, but instead we will use various properties and methods of calculation that are proved in the appendix.

Example 12.3. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have the familiar formula $\det(A) = ad - bc$. For a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ we have

$$\det(A) = aei - afh - bdi + bfg + cdh - ceg.$$

For more details of how this matches up with Definition 12.1, see Examples B.8 and B.9 in Appendix B.

Example 12.4. Let A be an $n \times n$ matrix.

(a) If all the entries below the diagonal are zero, then the determinant is just the product of the diagonal entries:

$\det(A) = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$. For example, we have

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1 \times 5 \times 8 \times 10 = 400.$$

(b) Similarly, if all the entries above the diagonal are zero, then the determinant is just the product of the diagonal entries.
(c) In particular, if A is a diagonal matrix (so all entries off the diagonal are zero) then both (a) and (b) apply and we have $\det(A) = \prod_{i=1}^{n} a_{ii}$.
(d) In particular, we have $\det(I_n) = 1$.

All these facts follow from Proposition B.11.

Example 12.5. If any row or column of A is zero, then $\det(A) = 0$. □

Proposition 12.6. *The determinants of elementary matrices are* $\det(D_p(\lambda)) = \lambda$ *and* $\det(E_{pq}(\mu)) = 1$ *and* $\det(F_{pq}) = -1$.

Proof. See Proposition B.12. □

Proposition 12.7. *For any square matrix A, we have* $\det(A^T) = \det(A)$.

Proof. See Corollary B.16. □

Theorem 12.8. *If A and B are $n \times n$ matrices, then* $\det(AB) = \det(A)\det(B)$.

Proof. See Theorem B.17. □

Method 12.9. Let A be an $n \times n$ matrix. We can calculate $\det(A)$ by applying row operations to A until we reach a matrix B for which we know $\det(B)$, keeping track of some factors as we go along.

(a) Every time we multiply a row by a number λ, we record the factor λ.
(b) Every time we exchange two rows, we record the factor -1.

Let μ be the product of these factors: then $\det(A) = \det(B)/\mu$.

In this method, we can if we wish continue the row reduction until we reach a matrix B in RREF. Then B will either be the identity (in which case $\det(B) = 1$ and $\det(A) = 1/\mu$) or B will have a row of zeros (in which case $\det(A) = \det(B) = 0$). However, it will often be more efficient to stop the row-reduction at an earlier stage.

Example 12.10. Take $A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. We can row-reduce this as follows:

$$A \to \begin{bmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{8}} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = B$$

The steps are as follows:

(1) We add multiples of row 4 to the other rows. This does not give any factor for the determinant.
(2) We multiply each of the first three rows by $\frac{1}{2}$, which gives an overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$. We have written this on the relevant arrow.
(3) We subtract row 1 from row 2.
(4) We subtract row 3 from row 2.
(5) We exchange rows 2 and 4, giving a factor of -1.
(6) We exchange rows 1 and 2, giving another factor of -1.

The final matrix B is upper-triangular, so the determinant is just the product of the diagonal entries, which is $\det(B) = 2$. The product of the factors is $\mu = 1/8$, so $\det(A) = \det(B)/\mu = 16$.

Example 12.11. Take $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$. We can start row-reducing as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

(In the first step we subtract row 1 from each of the other rows, and in the second we subtract multiples of row 2 from rows 3 and 4.) As B has two rows of zeros, we see that $\det(B) = 0$. The method therefore tells us that $\det(A) = 0$ as well.

Definition 12.12. Let A be an $n \times n$ matrix, and let p and q be integers with $1 \leq p, q \leq n$.

(a) We let M_{pq} be the matrix obtained by deleting the p'th row and the q'th column from A. This is a square matrix of shape $(n-1)\times(n-1)$.
(b) We put $m_{pq} = \det(M_{pq})$.
(c) We let $\mathrm{adj}(A)$ denote the $n\times n$ matrix with entries $\mathrm{adj}(A)_{qp} = (-1)^{p+q} m_{pq}$. (Note that we have qp on the left and pq on the right here.)

We call the matrices M_{pq} the *minor matrices* for A, and the numbers m_{pq} the *minor determinants*. The matrix $\mathrm{adj}(A)$ is called the *adjugate* (or sometimes the *classical adjoint*) of A.

Proposition 12.13. *The determinant* $\det(A)$ *can be "expanded along the first row", in the sense that*

$$\det(A) = a_{11}m_{11} - a_{12}m_{12} + \cdots \pm a_{1n}m_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} m_{1j}.$$

More generally, it can be expanded along the p'th row for any p, in the sense that

$$\det(A) = \sum_{j=1}^{n} (-1)^{p+j} a_{pj} m_{pj}.$$

Similarly, it can be expanded down the q'th column for any q, in the sense that

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+q} a_{iq} m_{iq}.$$

Example 12.14. Consider the matrix

$$A = \begin{bmatrix} a & 0 & b & c \\ 0 & 0 & 0 & d \\ e & f & g & h \\ i & 0 & j & k \end{bmatrix}.$$

We expand along the second row, to take advantage of the fact that that row contains many zeros. This gives

$$\det(A) = (-1)^{2+1}0 \times m_{21} + (-1)^{2+2}0 \times m_{22} +$$

$$(-1)^{2+3}0 \times m_{23} + (-1)^{2+4}d \times m_{24} = dm_{24}.$$

We will write B for the minor matrix obtained by deleting from A the row and column containing d, which leaves

$$B = \begin{bmatrix} a & 0 & b \\ e & f & g \\ i & 0 & j \end{bmatrix}.$$

By definition, we have $m_{24} = \det(B)$, so $\det(A) = d\,\det(B)$.

To evaluate $\det(B)$, we expand down the middle column. Again, there is only one nonzero term. The number f appears in the $(2,2)$ slot, so it comes with a sign $(-1)^{2+2} = +1$. The complementary minor matrix, obtained by deleting the middle row and the middle column, is $C = \begin{bmatrix} a & b \\ i & j \end{bmatrix}$. We thus have

$$\det(B) = (-1)^{2+2} f\,\det(C) = f(aj - bi),$$

and so $\det(A) = df(aj - bi) = adfj - bdfi$.

Example 12.15. Consider the matrix

$$U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Expanding along the top row gives

$$\det(U) = \det(V_1) - 0 \times \det(V_2) + \det(V_3) - 0 \times \det(V_4),$$

where

$$V_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad V_4 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

In V_1 the first and last rows are the same, so after a single row operation we have a row of zeros, which means that $\det(V_1) = 0$. We need not work out $\det(V_2)$ and $\det(V_4)$ because they will be multiplied by zero anyway. This just leaves $\det(U) = \det(V_3)$, which we can expand along the top row again:

$$\det(V_3) = 0 \times \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0 - 1 + 0 = -1.$$

We conclude that $\det(U) = -1$. It is a good exercise to obtain the same result by row-reduction.

Theorem 12.16. *Let A be an $n \times n$ matrix.*

(a) *If $\det(A) \neq 0$ then we can define $A^{-1} = \operatorname{adj}(A)/\det(A)$, and this matrix satisfies $A^{-1}A = I_n$ and also $AA^{-1} = I_n$. In other words, A^{-1} is an inverse for A. In this case, the only vector $v \in \mathbb{R}^n$ satisfying $Av = 0$ is the vector $v = 0$.*

(b) *If $\det(A) = 0$ then there is no matrix B with $BA = I_n$, and similarly there is no matrix C with $AC = I_n$. In other words, A has no inverse. Moreover, in this case there exists a nonzero vector v with $Av = 0$.*

Proof. See Theorem B.27. □

While the above formula for A^{-1} is theoretically appealing, it is not usually very efficient, especially for large matrices. Method 11.11 is generally better.

Example 12.17. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the minor matrices are 1×1 matrices or in other words just numbers, so $m_{ij} = \det(M_{ij}) = M_{ij}$. Specifically, we have

$$
\begin{aligned}
m_{11} &= d & m_{12} &= c \\
m_{21} &= b & m_{22} &= a
\end{aligned}
$$

so

$$
\operatorname{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} \\ -m_{12} & +m_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},
$$

so

$$
A^{-1} = \operatorname{adj}(A)/\det(A) = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
$$

Of course this formula is well known and can be derived in more elementary ways.

Example 12.18. Consider an upper triangular matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. This has $\det(A) = 1$ by Example 12.4. The minor determinants are

$$m_{11} = \det \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1$$

$$m_{12} = \det \begin{bmatrix} 0 & c \\ 0 & 1 \end{bmatrix} = 0$$

$$m_{13} = \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{21} = \det \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = a$$

$$m_{22} = \det \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = 1$$

$$m_{23} = \det \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} = 0$$

$$m_{31} = \det \begin{bmatrix} a & b \\ 1 & c \end{bmatrix} = ac - b$$

$$m_{32} = \det \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} = c$$

$$m_{33} = \det \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1.$$

We now form the adjugate, remembering that $\text{adj}(A)_{ij} = (-1)^{i+j} m_{ji}$ (with the indices backwards).

$$\text{adj}(A) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

We also have $A^{-1} = \text{adj}(A)/\det(A)$ but $\det(A) = 1$ so $A^{-1} = \text{adj}(A)$. Note that this is the same answer as we obtained in Example 11.12.

Example 12.19. Consider the matrix

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding minor matrices are as follows:

$$M_{11} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{13} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_{14} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{23} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_{24} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{33} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_{34} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{41} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad M_{42} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{43} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_{44} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

By inspection, each of these matrices is either upper triangular or lower triangular, so in each case the determinant is the product of the diagonal entries. This gives

$$\begin{array}{llll}
m_{11} = 1 & m_{12} = 0 & m_{13} = 0 & m_{14} = 0 \\
m_{21} = 1 & m_{22} = 1 & m_{23} = 0 & m_{24} = 0 \\
m_{31} = 1 & m_{32} = 1 & m_{33} = 1 & m_{34} = 0 \\
m_{41} = 1 & m_{42} = 1 & m_{43} = 1 & m_{44} = 1
\end{array}$$

and thus

$$\operatorname{adj}(P) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} & -m_{41} \\ -m_{12} & +m_{22} & -m_{32} & +m_{42} \\ +m_{13} & -m_{23} & +m_{33} & -m_{43} \\ -m_{14} & +m_{24} & -m_{34} & +m_{44} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As P is upper triangular it is easy to see that $\det(P) = 1$ and so P^{-1} is the same as $\operatorname{adj}(P)$.

13. Eigenvalues and eigenvectors

Definition 13.1. Let A be an $n \times n$ matrix, and let λ be a real number. A λ-*eigenvector* for A is a nonzero vector $v \in \mathbb{R}^n$ with the property that $Av = \lambda v$. We say that λ is an *eigenvalue* of A if there exists a λ-eigenvector for A.

Remark 13.2. Note that the whole theory of eigenvalues and eigenvectors only makes sense for *square* matrices. If A is not square then the vectors Av and λv will have different dimensions, so we cannot ask for them to be equal.

Remark 13.3. Note that if $Av = \lambda v$ then Av points in exactly the same direction as v (if $\lambda > 0$) or in the opposite direction (if

$\lambda < 0$) or $Av = 0$ (if $\lambda = 0$). All three of these are rather special situations that are unlikely to hold for a randomly chosen vector v.

Remark 13.4. We will mostly focus on real eigenvalues and eigenvectors. However, you should be aware that many aspects of the theory work better if we also consider complex eigenvalues and eigenvectors (even if the entries in the matrix are all real).

Example 13.5. Consider the case

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have

$$Aa = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2a$$

$$Ab = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0b$$

so a is a 2-eigenvector and b is a 0-eigenvector, so 2 and 0 are eigenvalues. Now consider a number $\lambda \neq 0, 2$. Suppose we have $A\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \lambda \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ or equivalently $\begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$. This means that $x + y = \lambda x$ and $x + y = \lambda y$. Subtracting these gives $\lambda(x - y) = 0$ but $\lambda \neq 0$ so $x = y$. Substituting this back in gives $2x = \lambda x$ or $(\lambda - 2)x = 0$, and $\lambda \neq 2$ so we can divide by $\lambda - 2$ to get $x = 0$. We have also seen that $y = x$, so $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. This means that there is no λ-eigenvector, so λ is not an eigenvalue. It follows that 0 and 2 are the *only* eigenvalues of A.

We can also reach the same conclusion by row-reduction. To solve the equation $Av = \lambda v$ we need to row-reduce the matrix $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$. The first step is to subtract $1 - \lambda$

times row 2 from row 1, giving $\begin{bmatrix} 0 & 1-(1-\lambda)^2 \\ 1 & 1-\lambda \end{bmatrix}$. Here we have

$$1-(1-\lambda)^2 = 2\lambda - \lambda^2 = \lambda(2-\lambda),$$

which is nonzero because $\lambda \notin \{0,2\}$. We can thus divide the first row by this to get $\begin{bmatrix} 0 & 1 \\ 1 & 1-\lambda \end{bmatrix}$, and a couple more steps reduce this to the identity matrix I_2. This means that the equation $(A-\lambda I_2)v = 0$ has the same solutions as the equation $I_2 v = 0$, namely $v = 0$. As there are no nonzero solutions, we see that λ is not an eigenvalue.

Example 13.6. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

and the vectors

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix} \qquad d = \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix}.$$

We have

$$Ad = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 36 \\ 12 \end{bmatrix} = 4d,$$

which means that d is a 4-eigenvector for A, and 4 is an eigenvalue of A. Equally direct calculation shows that $Aa = a$ and $Ab = 2b$ and $Ac = 3c$, so a, b and c are also eigenvectors, and 1, 2 and 3 are also eigenvalues of A. Using the general theory that we will discuss below, we can show that

(a) The only 1-eigenvectors are the nonzero multiples of a.
(b) The only 2-eigenvectors are the nonzero multiples of b.
(c) The only 3-eigenvectors are the nonzero multiples of c.
(d) The only 4-eigenvectors are the nonzero multiples of d.
(e) There are no more eigenvalues: if λ is a real number other than 1, 2, 3 and 4, then the equation $Av = \lambda v$ has $v = 0$ as the only solution, so there are no λ-eigenvectors.

Remark 13.7. You can ask Maple to calculate the eigenvalues of A by entering `Eigenvalues(A)`. If A is an $n \times n$ matrix, then this will give a column vector of length n whose entries are the eigenvalues. One can also enter `Eigenvectors(A)` to find the eigenvectors. In typical cases this will give a pair of things, the first of which is the vector of eigenvalues, and the second of which is a square matrix whose columns are the corresponding eigenvectors. However, there are some subtleties about what happens if some eigenvalues are repeated; we postpone any discussion of this. Python equivalents are `A.eigenvals()` and `A.eigenvects()`.

Definition 13.8. Let A be an $n \times n$ matrix. We define

$$\chi_A(t) = \det(A - t\, I_n)$$

(where I_n is the identity matrix, as usual). This is called the *characteristic polynomial* of A.

Note that some authors have a slightly different convention, and define the characteristic polynomial to be $\det(tI_n - A)$. In particular, if you enter `CharacteristicPolynomial(A,t)` in Maple you will get $\det(tI_n - A)$. It is easy to relate these conventions, because $\det(tI_n - A) = (-1)^n \det(A - t\, I_n)$. Note also that in Maple we need to specify the variable t as well as the matrix A. If you just enter `CharacteristicPolynomial(A)` you will get an error. The corresponding syntax in Python is

`A.charpoly(t)`; this calculates $\det(tI_n - A)$, so we do not need to correct the sign.

Example 13.9. For any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have $A - tI_2 = \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix}$ so

$$\chi_A(t) = \det \begin{bmatrix} a-t & b \\ c & d-t \end{bmatrix} =$$

$$(a-t)(d-t) - bc = t^2 - (a+d)t + (ad - bc).$$

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we have

$$\chi_A(t) = t^2 - (1+4)t + (1 \times 4 - 2 \times 3) = t^2 - 5t - 2.$$

Example 13.10. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} \chi_A(t) &= \det \begin{bmatrix} -1-t & 0 & 1 \\ 1 & -1-t & 0 \\ 0 & 1 & -1-t \end{bmatrix} \\ &= (-1-t)\det \begin{bmatrix} -1-t & 0 \\ 1 & -1-t \end{bmatrix} \\ &\quad - 0 \det \begin{bmatrix} 1 & 0 \\ 0 & -1-t \end{bmatrix} \\ &\quad + \det \begin{bmatrix} 1 & -1-t \\ 0 & 1 \end{bmatrix} \\ &= (-1-t)^3 - 0 + 1 = 1 - (1+t)^3 = -3t - 3t^2 - t^3. \end{aligned}$$

Note that λ is an eigenvalue of A if and only if there is a nonzero solution to the equation $Au = \lambda u$, or equivalently the equation $(A - \lambda I_n)u = 0$. Moreover, if B is an $n \times n$ matrix then the equation $Bu = 0$ has a nonzero solution iff B is not invertible iff $\det(B) = 0$ (by Theorem 12.16). Using this, we obtain the following result:

Theorem 13.11. *Let A be a square matrix. Then the eigenvalues of A are the roots of the characteristic polynomial $\chi_A(t)$.*

Corollary 13.12. *For any $n \times n$ matrix A we have $\chi_A(t) = \chi_{A^T}(t)$, so A and A^T havs the same eigenvalues.*

Proof. As $I_n^T = I_n$, we have $\chi_{A^T}(t) = \det(A^T - tI_n) = \det((A - tI_n)^T)$, which is the same as $\det(A - tI_n)$ by Proposition 12.7. $\square$

Example 13.13. We will find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}
\chi_A(t) &= \det \begin{bmatrix} -1-t & 1 & 0 \\ -1 & -t & 1 \\ -1 & 0 & -t \end{bmatrix} \\
&= (-1-t)\det \begin{bmatrix} -t & 1 \\ 0 & -t \end{bmatrix} - \det \begin{bmatrix} -1 & 1 \\ -1 & -t \end{bmatrix} + 0 \det \begin{bmatrix} -1 & -t \\ -1 & 0 \end{bmatrix} \\
&= -t^2(1+t) - (t+1) + 0 = -(1+t^2)(1+t).
\end{aligned}$$

As $1 + t^2$ is always positive, the only way $-(1+t^2)(1+t)$ can be zero is if $t = -1$. Thus, the only real eigenvalue of A is -1. (There are also complex eigenvalues, namely i and $-i$, but we will only consider the real ones here.) When $\lambda = -1$ we have

$$A - \lambda I_3 = A + I_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

To find an eigenvector of eigenvalue -1, we need to solve the equation $(A + I_3)u = 0$, or equivalently

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or equivalently

$$y = 0 \qquad -x + y + z = 0 \qquad -x + z = 0.$$

These equations reduce to $x = z$ with $y = 0$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

This means that the (-1)-eigenvectors are just the nonzero multiples of the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

In the above example we solved the equation $(A + I_3)u = 0$ in a rather *ad hoc* way. A more systematic approach would be use row reduction, as follows.

Method 13.14. Suppose we have an $n \times n$ matrix A, and we want to find the eigenvalues and eigenvectors.

(a) Calculate the characteristic polynomial $\chi_A(t) = \det(A - tI_n)$.
(b) Find all the real roots of $\chi_A(t)$, and list them as $\lambda_1, \dots, \lambda_k$. These are the eigenvalues of A.

(c) For each eigenvalue λ_i, row reduce the matrix $A - \lambda_i I_n$ to get a matrix B.

(d) Read off solutions to the equation $Bu = 0$ as in Method 5.4 These are the λ_i-eigenvectors of the matrix A.

Remark 13.15. When trying to carry out this method, you might find at step (c) that B is the identity matrix, so the equation $Bu = 0$ in step (d) just gives $u = 0$. You might then be tempted to say that 0 is an eigenvector of A. This is incorrect: the zero vector is, by definition, not an eigenvector. If you find yourself in this situation, then **you have made a mistake**. Either λ_i is not really an eigenvalue, or there is an error in your row reduction. You should go back and check your work.

Example 13.16. Consider the matrix

$$A = \begin{bmatrix} 16 & 2 & 1 & 1 \\ 2 & 16 & 1 & 1 \\ 1 & 1 & 16 & 2 \\ 1 & 1 & 2 & 16 \end{bmatrix}$$

We will take it as given here that

$$\chi_A(t) = (t-14)^2(t-16)(t-20).$$

The eigenvalues of A are the roots of $\chi_A(t)$, namely 14, 16 and 20. To find the eigenvectors of eigenvalue 14, we write down the matrix $A - 14I_4$ and row-reduce it to get a matrix B as follows:

$$A{-}14I_4 = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -3 & -3 \\ 0 & 0 & -3 & -3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a + b = c + d = 0$, so $a = -b$ and $c = -d$ (with b and d

arbitrary), so

$$u = \begin{bmatrix} -b \\ b \\ -d \\ d \end{bmatrix}$$

for some $b, d \in \mathbb{R}$. The eigenvectors of eigenvalue 14 are precisely the nonzero vectors of the above form. (Recall that eigenvectors are nonzero, by definition.)

Remark 13.17. Using Maple, we find that one eigenvalue of the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

is

$$\lambda = -\frac{1}{2} + \sqrt{\frac{X}{24}} + \sqrt{\frac{5}{4} - \frac{X}{24} + \sqrt{\frac{27}{2X}}}$$

where $X = 10 + Y + 28/Y$ and $Y = (892 + 36\sqrt{597})^{1/3}$. This level of complexity is quite normal, even for matrices whose entries are all 0 or ± 1. Most examples in this course are carefully constructed to have simple eigenvalues and eigenvectors, but you should be aware that this is not typical. The methods that we discuss will work perfectly well for all matrices, but in practice we need to use computers to do the calculations. Also, it is rarely useful to work with exact expressions for the eigenvalues when they are as complicated as those above. Instead we should use the numerical approximation $\lambda \simeq 1.496698205$.

Example 13.18. Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

We will take it as given that the characteristic polynomial is

$$\chi_A(t) = (t+1)(t+2)(t-2)(t-4),$$

so the eigenvalues are -1, -2, 2 and 4. To find the eigenvectors of eigenvalue 2, we write down the matrix $A - 2I_4$ and row-reduce it to get a matrix B in RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

If we write $u = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$, then the equation $Bu = 0$ just gives $a = b - c = d = 0$, so

$$u = \begin{bmatrix} 0 \\ c \\ c \\ 0 \end{bmatrix} = c \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

for some $c \in \mathbb{R}$. The eigenvectors of eigenvalue 2 are precisely the nonzero vectors of the above form. In particular, the vector $\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$ is an eigenvector of eigenvalue 2.

There is an important connecton between eigenvectors and linear independence, as follows.

Proposition 13.19. *Let A be a $d \times d$ matrix, and let $v_1, \ldots, v_n$ be eigenvectors of A. This means that each v_i is nonzero and there is a scalar λ_i such that $Av_i = \lambda_i v_i$. Suppose also that the* *eigenvalues $\lambda_1, \dots, \lambda_n$ are all different. Then the list $v_1, \dots, v_n$ is linearly independent.*

Proof. We first consider the case $n = 2$, where we just have an eigenvector v_1 with eigenvalue λ_1, and another eigenvector v_2 with a different eigenvalue λ_2 (so $\lambda_1 - \lambda_2 \neq 0$). Suppose we have a linear relation

$$\alpha_1 v_1 + \alpha_2 v_2 = 0. \tag{P}$$

We now multiply both sides of this vector equation by the matrix $A - \lambda_2 I$. Because $Av_i = \lambda_i v_i$ we have $(A - \lambda_2 I)v_1 = (\lambda_1 - \lambda_2)v_1$ and $(A - \lambda_2)v_2 = 0$, so we get

$$(\lambda_1 - \lambda_2)\alpha_1 v_1 = 0.$$

As the number $\lambda_1 - \lambda_2$ and the vector v_1 are nonzero, we can conclude that $\alpha_1 = 0$. If we instead multiply equation (P) by $A - \lambda_1 I$ we get

$$(\lambda_2 - \lambda_1)\alpha_2 v_2 = 0.$$

As the number $\lambda_2 - \lambda_1$ and the vector v_2 are nonzero, we can conclude that $\alpha_2 = 0$. We have now seen that $\alpha_1 = \alpha_2 = 0$, so the relation (P) is the trivial relation. As this works for any linear relation between v_1 and v_2, we see that these vectors are linearly independent.

Now consider instead the case $n = 3$. Suppose we have a linear relation

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0. \tag{Q}$$

We multiply this by $A - \lambda_3 I$, remembering that $(A - \lambda_3 I)v_i = (\lambda_i - \lambda_3)v_i$, which is zero for $i = 3$. This gives

$$(\lambda_1 - \lambda_3)\alpha_1 v_1 + (\lambda_2 - \lambda_3)\alpha_2 v_2 = 0. \tag{R}$$

We now multiply (R) by $A - \lambda_2 I$. The term involving v_2 goes away because $(A - \lambda_2)v_2 = 0$, so we are just left with

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\alpha_1 v_1 = 0.$$

By assumption, the eigenvalues λ_1, λ_2 and λ_3 are all different, so the number $(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)$ is nonzero. The vector v_1 is also nonzero (because eigenvectors are nonzero by definition) so we can conclude that $\alpha_1=0$. Similarly:

- If we multiply (Q) by $(A-\lambda_1 I)(A-\lambda_3 I)$ then the first and third terms go away and we are left with $(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\alpha_2 v_2=0$, which implies that $\alpha_2=0$.
- If we multiply (Q) by $(A-\lambda_1 I)(A-\lambda_2 I)$ then the first and second terms go away and we are left with $(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)\alpha_3 v_3=0$, which implies that $\alpha_3=0$.

We have now seen that $\alpha_1=\alpha_2=\alpha_3=0$, so the relation (Q) is the trivial relation. As this works for any linear relation between v_1, v_2 and v_3, we see that these vectors are linearly independent.

The general case works the same way. Suppose we have a linear relation

$$\alpha_1 v_1+\cdots+\alpha_n v_n=0. \tag{S}$$

For any index k, we can multiply by all the matrices $A-\lambda_i I$ for $i\neq k$. This makes all the terms go away except for the k'th term, leaving

$$\left(\prod_{i\neq k}(\lambda_i-\lambda_k)\right)\alpha_k v_k=0.$$

As all the eigenvalues λ_j are assumed to be different, the product $\prod_{i\neq k}(\lambda_i-\lambda_k)$ is nonzero, so we can divide by it to get $\alpha_k v_k=0$. As $v_k\neq 0$ this gives $\alpha_k=0$. This works for all k, so (S) is the trivial relation. $\square$

Remark 13.20. We can reorganise the proof as an induction on the number n of eigenvectors. The case $n=1$ is trivial, because if we take a nonzero vector v_1 then the list consisting of just v_1 is always independent. Suppose we know that the proposition is true for any list of $n-1$ eigenvectors with distinct eigenvalues. Suppose we have a list $v_1, \ldots, v_n$ as in the Proposition, and a linear relation

$$\alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} + \alpha_n v_n = 0. \tag{P}$$

We multiply by $A - \lambda_n I$ and note that the last term goes away, leaving

$$(\lambda_1 - \lambda_n)\alpha_1 v_1 + \cdots + (\lambda_{n-1} - \lambda_n)\alpha_{n-1} v_{n-1} = 0. \tag{Q}$$

This is a linear relation on the list $v_1, \ldots, v_{n-1}$. However, that list is independent by the induction hypothesis, so relation (Q) must be the trivial relation, which means that

$$(\lambda_1 - \lambda_n)\alpha_1 = \cdots = (\lambda_{n-1} - \lambda_n)\alpha_{n-1} = 0.$$

As all the eigenvalues are assumed to be distinct, we can divide by the numbers $\lambda_i - \lambda_n$ to get

$$\alpha_1 = \cdots = \alpha_{n-1} = 0.$$

Substituting this into (P) gives $\alpha_n v_n = 0$ but $v_n \neq 0$ so $\alpha_n = 0$. Thus, relation (P) is the trivial relation, as required. This completes the induction step, so the Proposition holds for all n.

Remark 13.21. Proposition 13.19 can be generalised as follows. Suppose we have:

- A $d \times d$ matrix A
- A list $\lambda_1, \ldots, \lambda_r$ of distinct eigenvalues
- A linearly independent list $\mathcal{V}_1 = (v_{1,1}, \ldots, v_{1,h_1})$ of eigenvectors, all with eigenvalue λ_1
- A linearly independent list $\mathcal{V}_2 = (v_{2,1}, \ldots, v_{2,h_2})$ of eigenvectors, all with eigenvalue λ_2
- $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
- A linearly independent list $\mathcal{V}_r = (v_{r,1}, \ldots, v_{r,h_r})$ of eigenvectors, all with eigenvalue λ_r

We can then combine the lists $\mathcal{V}_1, \dots, \mathcal{V}_r$ into a single list

$$\mathcal{W} = (v_{1,1}, \cdots, v_{1,h_1}, v_{2,1}, \cdots, v_{2,h_2}, \cdots, v_{r,1}, \cdots, v_{r,h_r}).$$

One can then show that the combined list $\mathcal{W}$ is linearly independent.

Proposition 13.22. *Let A be an $n \times n$ matrix, and suppose that A has eigenvalues $\lambda_1, \dots, \lambda_n$ which are all different. Let u_i be an eigenvector for A with eigenvalue λ_i. Then the list $\mathcal{U} = u_1, \dots, u_n$ is a basis for $\mathbb{R}^n$.*

Proof. Proposition 13.19 tells us that $\mathcal{U}$ is linearly independent. We therefore have a linearly independent list of length n in $\mathbb{R}^n$, so it must be a basis by Proposition 10.12. □

Example 13.23. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. It is easy to see that $\chi_A(t) = \det(A-tI) = (1-t)(2-t)(3-t)$, so the eigenvalues are 1, 2 and 3. Suppose we have eigenvectors u_1, u_2 and u_3, where u_k has eigenvalue k. The Proposition then tells us that the list u_1, u_2, u_3 is automatically a basis for $\mathbb{R}^3$. We can find the eigenvectors explicitly using Method 13.14. The relevant row-reductions and eigenvectors are shown below:

$$\begin{array}{rclcll}
A - I & = & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} & \to & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
A - 2I & = & \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} & \to & \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
A - 3I & = & \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} & \to & \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} & u_3 = \begin{bmatrix} 3/2 \\ 2 \\ 1 \end{bmatrix}
\end{array}$$

Proposition 13.22 now tells us that the vectors u_k form a basis. This can be checked more directly by forming the matrix $U = [u_1|u_2|u_3]$ and row-reducing it:

$$U = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

As U reduces to the identity matrix, Method 10.5 tells us that the columns u_k form a basis, as expected.

Example 13.24. Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The characteristic polynomial is $\chi_A(t) = \det \begin{bmatrix} -t & 1 \\ -1 & -t \end{bmatrix} = t^2+1$. For all $t \in \mathbb{R}$ we have $t^2 + 1 \geq 1 > 0$, so the characteristic polynomial has no real roots, so there are no real eigenvalues or eigenvectors. However, if we use complex numbers we can say that the eigenvalues are i and $-i$, with corresponding eigenvectors

$$u_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix},$$

which form a basis for $\mathbb{C}^2$.

Remark 13.25. Examples 13.23 and 13.24 illustrate the typical case. If we pick an $n \times n$ matrix at random, it will usually have n different eigenvalues (some of which will usually be complex), and so the corresponding eigenvectors will form a basis for $\mathbb{C}^n$. However, there are some exceptions, as we will see in the next two examples. Such exceptions usually arise because of some symmetry or other interesting feature of the problem that gives rise to the matrix.

Example 13.26. Consider the matrix $A = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{bmatrix}$. We have $\chi_A(t) = (5-t)^3$, so the only eigenvalue is 5. The eigenvectors are the solutions of $(A - 5I)u = 0$. If $u = \begin{bmatrix} x & y & z \end{bmatrix}^T$ then this reduces to $5y = 5z = 0$ so $y = z = 0$ and u is a multiple of the vector $a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This means that any two eigenvectors are multiples of each other, and so are linearly dependent. Thus, we cannot find a basis consisting of eigenvectors.

Example 13.27. Consider the matrix $\begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$. The characteristic polynomial is

$$\begin{aligned}
\chi_A(t) &= \det \begin{bmatrix} -t & 0 & 5 \\ 0 & 5-t & 0 \\ 5 & 0 & -t \end{bmatrix} \\
&= -t \det \begin{bmatrix} 5-t & 0 \\ 0 & -t \end{bmatrix} + 5 \det \begin{bmatrix} 0 & 5-t \\ 5 & 0 \end{bmatrix} \\
&= -125 + 25t + 5t^2 - t^3 = -(t-5)(t^2-25) \\
&= -(t-5)(t-5)(t+5) \\
&= -(t-5)^2(t+5).
\end{aligned}$$

This shows that the only eigenvalues are 5 and -5 (and we would not get any more if we allowed complex numbers). Thus, Proposition 13.22 cannot be used here, because we do not have three different eigenvalues. Nonetheless, in this example it turns out that there is still a basis consisting of eigenvectors, even though Proposition 13.22 cannot be used to prove that fact. Indeed, we can take

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

It is straightforward to check that $Au_1 = 5u_1$ and $Au_2 = 5u_2$ and $Au_3 = -5u_3$, so the u_i are eigenvectors with eigenvalues 5, 5 and -5 respectively. It is also easy to see that the vectors u_i form a basis, either by row-reducing the matrix $[u_1|u_2|u_3]$ or by noting that an arbitrary vector $v = \begin{bmatrix} x & y & z \end{bmatrix}^T$ can be expressed in a unique way as a linear combination of the u_i, namely

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (x+z)/2 \\ 0 \\ (x+z)/2 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} (x-z)/2 \\ 0 \\ (z-x)/2 \end{bmatrix} =$$

$$\frac{x+z}{2}u_1 + yu_2 + \frac{x-z}{2}u_3.$$

14. Diagonalisation

We next show how eigenvectors can be used to transform various problems about square matrices to problems about diagonal matrices, which are generally much easier.

Definition 14.1. We write $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ for the $n \times n$ matrix such that the entries on the diagonal are $\lambda_1, \dots, \lambda_n$ and the entries off the diagonal are zero.

Example 14.2. $\mathrm{diag}(5, 6, 7, 8) = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$

Definition 14.3. Let A be an $n \times n$ matrix. To *diagonalise* A means to give an invertible matrix U and a diagonal matrix D such that $U^{-1}AU = D$ (or equivalently $A = UDU^{-1}$). We say that A is *diagonalisable* if there exist matrices U and D with these properties.

Proposition 14.4. *Suppose we have a basis $u_1, \dots, u_n$ for $\mathbb{R}^n$ such that each vector u_i is an eigenvector for A, with eigenvalue λ_i say. Put $U = [u_1|\cdots|u_n]$ and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Then $U^{-1}AU = D$, so we have a diagonalisation of A. Moreover, every diagonalisation of A occurs in this way.*

To prove this, we need two standard facts about matrix multiplication.

Lemma 14.5. *Let A and U be $n \times n$ matrices, let $\lambda_1, \dots, \lambda_n$ be real numbers, and put $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Let $u_1, \dots, u_n$ be the columns of U. Then*

$$AU = \left[\begin{array}{c|c|c} Au_1 & \cdots & Au_n \end{array}\right]$$

$$UD = \left[\begin{array}{c|c|c} \lambda_1 u_1 & \cdots & \lambda_n u_n \end{array}\right].$$

Proof. First let the rows of A be $a_1^T, \dots, a_n^T$. By the definition of matrix multiplication, we have

$$AU = \begin{bmatrix} a_1.u_1 & \cdots & a_1.u_n \\ \cdots & \cdots & \cdots \\ a_n.u_1 & \cdots & a_n.u_n \end{bmatrix}$$

The p'th column is $\begin{bmatrix} a_1.u_p \\ \vdots \\ a_n.u_p \end{bmatrix}$, and this is just the definition of Au_p. In other words, we have

$$AU = \left[\begin{array}{c|c|c} Au_1 & \cdots & Au_n \end{array}\right]$$

as claimed. For the second claim, we will just give an example that makes the pattern clear. If $U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then

$$UD = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 a & \lambda_2 b & \lambda_3 c \\ \lambda_1 d & \lambda_2 e & \lambda_3 f \\ \lambda_1 g & \lambda_2 h & \lambda_3 i \end{bmatrix}$$

Everything in the first column gets multiplied by λ_1, everything in the second column gets multiplied by λ_2 and everything in the third column gets multiplied by λ_3. In other words, we have

$$\left[\begin{array}{c|c|c} u_1 & u_2 & u_3 \end{array}\right] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \left[\begin{array}{c|c|c} \lambda_1 u_1 & \lambda_2 u_2 & \lambda_3 u_3 \end{array}\right]$$

as claimed. $\square$

Proof of Proposition 14.4. First suppose we have a basis $u_1, \dots, u_n$ of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$. Put $U = [u_1|\cdots|u_n]$. As the columns of U form a basis for $\mathbb{R}^n$, Theorem 11.5 tells us that U is invertible. Moreover, Lemma 14.5 tells us that $AU = [Au_1|\cdots|Au_n]$, but u_i is assumed to be an eigenvector of eigenvalue λ_i, so $Au_i = \lambda_i u_i$, so $AU = [\lambda_1 u_1|\cdots|\lambda_n u_n]$. The other half of Lemma 14.5 tells us that UD can be described in the same way, so $AU = UD$. It follows that $U^{-1}AU = U^{-1}UD$, which is the same as D because $U^{-1}U = I_n$, so $U^{-1}AU = D$. Alternatively, from the equation $UD = AU$ we can also see that $UDU^{-1} = AUU^{-1} = A$.

Finally, this whole argument can be reversed in a straightforward way. Suppose we have an invertible matrix U and a diagonal matrix $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ such that $U^{-1}AU = D$. We let $u_1, \dots, u_n$ be the columns of U, and note that these form a basis for $\mathbb{R}^n$, because U is assumed to be invertible. The equation $U^{-1}AU = D$ implies that $UD = UU^{-1}AU = AU$. In the light of Lemma 14.5, we conclude that

$$\left[\begin{array}{c|c|c} \lambda_1 u_1 & \cdots & \lambda_n u_n \end{array}\right] = \left[\begin{array}{c|c|c} Au_1 & \cdots & Au_n \end{array}\right].$$

from this it is clear that $Au_i = \lambda_i u_i$ for all i, so u_i is an eigenvector of eigenvalue λ_i. □

We can now recycle some examples from Section 10.

Example 14.6. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. In Example 13.23 we saw that the following vectors are eigenvectors for A with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ and $\lambda_3 = 3$:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 3/2 \\ 2 \\ 1 \end{bmatrix}.$$

It follows that if we put

$$U = \left[\begin{array}{c|c|c} u_1 & u_2 & u_3 \end{array}\right] = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

then $A = UDU^{-1}$. To understand this explicitly we need to calculate U^{-1}. In Example 11.12, we saw that

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

By taking $a = 1$ and $b = 3/2$ and $c = 2$ we get

$$U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We thus have

$$DU^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}$$

$$UDU^{-1} = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

As expected, this is the same as A.

Example 14.7. In Example 13.24 we showed that the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ does not have any real eigenvalues or eigenvectors, but that over the complex numbers we have eigenvectors $u_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ with eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. We thus have a diagonalisation with $U = [u_1|u_2] = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ and $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Here the standard formula for the inverse of a 2×2 matrix gives

$$U^{-1} = \frac{1}{-2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix}.$$

This gives

$$UDU^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix} =$$

$$\begin{bmatrix} i & -i \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

As expected, this is the same as A.

Example 14.8. In Example 13.26 we saw that there is no basis of eigenvectors for the matrix $A = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{bmatrix}$. It follows that this matrix is not diagonalisable.

It is possible to understand non-diagonalisable matrices in great detail using the theory of *Jordan blocks*. However, we will not cover Jordan blocks in this course.

One point about eigenvectors and diagonalisation is that they make it easy to understand the powers A^k of a matrix A, which are required for many applications. First note that if v is an eigenvector of eigenvalue λ, we have

$$\begin{aligned} Av &= \lambda v \\ A^2 v &= A(Av) = A(\lambda v) = \lambda Av = \lambda^2 v \\ A^3 v &= A(A^2 v) = A(\lambda^2 v) = \lambda^2 Av = \lambda^3 v \end{aligned}$$

and so on, so $A^k v = \lambda^k v$ for all k. On the left hand side here we have powers of the matrix A, which are hard to calculate, but on the right hand side we just have powers of the number λ, which are easy. This can be souped up to give a formula for A^k, as follows.

Proposition 14.9. *Suppose we have a diagonalisation $A = UDU^{-1}$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ say. Then for all $k \geq 0$ we have $D^k = \mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k)$ and*

$$A^k = UD^kU^{-1} = U\ \mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k)\ U^{-1}.$$

Proof. First, we have

$$A^4 = (UDU^{-1})^4 = UDU^{-1}\,UDU^{-1}\,UDU^{-1}\,UDU^{-1}.$$

Now $U^{-1}U = I_n$ and multiplication by I_n does not do anything, so we can discard the terms $U^{-1}U$ leaving

$$A^4 = UDDDDU^{-1} = UD^4U^{-1}.$$

It is clear that the general case works the same way, so $A^k = UD^kU^{-1}$ for all k. One could give a more formal proof by induction if desired. Next, it is clear from the definition of matrix multiplication that

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,\mathrm{diag}(\mu_1, \ldots, \mu_n) = \mathrm{diag}(\lambda_1\mu_1, \ldots, \lambda_n\mu_n).$$

From this it is not hard to deduce that

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n)^k = \mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k).$$

(Again, a formal proof would go by induction on k.) □

Example 14.10. We will diagonalise the matrix

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

and thus find the powers A^k. As $A - tI_4$ is upper-triangular we see that the determinant is just the product of the diagonal terms. This gives

$$\chi_A(t) = \det(A - tI_4) = t^2(t-3)(t+3),$$

and it follows that the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 3$ and $\lambda_4 = -3$. Consider the vectors

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 2 \\ -3 \\ -6 \\ -9 \end{bmatrix}$$

It is straightforward to check that $Au_1 = Au_2 = 0$ and $Au_3 = 3u_3$ and $Au_4 = -3u_4$, so the vectors u_i are eigenvectors for A, with eigenvalues 0, 0, 3 and -3 respectively. (These vectors were found by row-reducing the matrices $A - \lambda_i I_4$ in the usual way, but we will not write out the details.) Now put

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ -3 & 0 & 0 & -6 \\ 0 & 0 & 0 & -9 \end{bmatrix} \qquad V = \frac{1}{9}\begin{bmatrix} 0 & 0 & -3 & 2 \\ 0 & -3 & 0 & 1 \\ 9 & 6 & 3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

The columns of U are $u_1, \dots, u_4$. One can check directly that $UV = I_4$, so U is invertible with $U^{-1} = V$. Using Theorem 11.5, it follows that the vectors u_i form a basis. We now see that $A = UDU^{-1}$, and thus that $A^k = UD^kU^{-1}$. More explicitly, we have

$$
\begin{aligned}
&A^k \\
=&\frac{1}{9}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ -3 & 0 & 0 & -6 \\ 0 & 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3^k & 0 \\ 0 & 0 & 0 & (-3)^k \end{bmatrix}\begin{bmatrix} 0 & 0 & -3 & 2 \\ 0 & -3 & 0 & 1 \\ 9 & 6 & 3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
=&3^{k-2}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \\ -3 & 0 & 0 & -6 \\ 0 & 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 9 & 6 & 3 & -2 \\ 0 & 0 & 0 & (-1)^{k+1} \end{bmatrix} \\
=&3^{k-2}\begin{bmatrix} 9 & 6 & 3 & -2(1+(-1)^k) \\ 0 & 0 & 0 & 3(-1)^k \\ 0 & 0 & 0 & 6(-1)^k \\ 0 & 0 & 0 & 9(-1)^k \end{bmatrix}
\end{aligned}
$$

Example 14.11. We will diagonalise the matrix

$$
A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}.
$$

We first need to calculate the characteristic polynomial, which is the determinant of the matrix

$$
B = A - tI_4 = \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix}.
$$

We will calculate this by Method 12.9 (which involves row-reducing B and keeping track of various factors that arise from the row operations). The row-reduction is as follows:

$$\begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 2 & 5 & 5-t & 2 \\ 2 & 2 & 2 & 2-t \end{bmatrix} \to \begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & t & -t & 0 \\ t & 0 & 0 & -t \end{bmatrix} \xrightarrow{1/t^2}$$

$$\begin{bmatrix} 2-t & 2 & 2 & 2 \\ 2 & 5-t & 5 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 4 & 4-t \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 10-t & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 4-t \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 10-t & 4 \\ 0 & 0 & 4 & 4-t \end{bmatrix}$$

(In the first step we subtracted row 1 from row 4, and row 2 from row 3. We then multiplied rows 3 and 4 by $1/t$, then subtracted multiples of rows 3 and 4 from rows 1 and 2. Finally we swapped rows 1 and 4, and then swapped rows 2 and 3.)

If we call the final matrix C, we can expand down the columns to get

$$\det(C) = \det \begin{bmatrix} 10-t & 4 \\ 4 & 4-t \end{bmatrix} = (10-t)(4-t) - 16 =$$

$$t^2 - 14t + 24 = (t-2)(t-12).$$

The product of the factors associated with the row-reduction steps is $\mu = (1/t^2).(-1)^2 = 1/t^2$. We therefore have

$$\chi_A(t) = \det(B) = \det(C)/\mu = (t-2)(t-12)t^2.$$

This means that the eigenvalues of A are 2, 12 and 0.

To find an eigenvector of eigenvalue 2 we need to row-reduce the matrix $A - 2I_4$, which is just the matrix B with $t = 2$. We can therefore substitute $t = 2$ in C and then perform a few more steps to complete the row-reduction.

$$A - 2I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvector $u_1 = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ of eigenvalue 2 must therefore satisfy $w - z = x + z/2 = y + z/2 = 0$, so $u_1 = z \begin{bmatrix} 1 & -1/2 & -1/2 & 1 \end{bmatrix}^T$, with z arbitrary. It will be convenient to take $z = 2$ so $u_1 = \begin{bmatrix} 2 & -1 & -1 & 2 \end{bmatrix}^T$.

Next, to find an eigenvector of eigenvalue 12 we need to row-reduce the matrix $A - 12I_4$, which is just the matrix B with $t = 12$. We can therefore substitute $t = 12$ in C and then perform a few more steps to complete the row-reduction.

$$A - 12I_4 \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we find that $u_2 = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T$ is an eigenvector of eigenvalue 12.

Finally, we need to find the eigenvectors of eigenvalue 0. Our reduction $B \to C$ involved division by t, so it is not valid in this case where $t = 0$. We must therefore start again and row-reduce the matrix $A - 0I_4 = A$ directly, but that is easy:

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 5 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that the eigenvectors of eigenvalue 0 are the vectors $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ with $w+z = x+y = 0$, and any such vector can be written in the form

$$v = \begin{bmatrix} w \\ x \\ -x \\ -w \end{bmatrix} = w \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

We therefore take

$$u_3 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T \qquad u_4 = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T$$

form a basis.

Now put

$$U = [u_1|u_2|u_3|u_4] = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 2 & 0 & -1 \\ 2 & 1 & -1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We next need to check that U is invertible and find the inverse. The systematic approach would be to row-reduce the matrix $[U|I_4]$ to get a matrix of the form $[I_4|V]$, then we could conclude that U is invertible with $U^{-1} = V$. We will not write this out, but instead we will just record the answer. Put

$$V = \frac{1}{10}\begin{bmatrix} 2 & -1 & -1 & 2 \\ 1 & 2 & 2 & 1 \\ 5 & 0 & 0 & -5 \\ 0 & 5 & -5 & 0 \end{bmatrix}.$$

One can then check by just multiplying out that $UV = I_4$, so $U^{-1} = V$. We therefore have a diagonalisation $A = UDU^{-1} = UDV$,

15. Differential equations

In this section we will explain how eigenvectors can be used to solve certain systems of coupled differential equations (of a type that are common in applications). First suppose that x is a function of t, and satisfies the equation

$$\dot{x} = ax$$

for some constant a. It is standard, and easy to check, that we must have $x = e^{at}c$ for some constant c. When $t = 0$ we have $e^{at} = 1$ and so $x = c$. In other words, c is just the initial value of x at $t = 0$.

Now suppose we have variables x_1, x_2 and x_3 satisfying equations of the same type:

$$\begin{aligned} \dot{x}_1 &= a_1\, x_1 \\ \dot{x}_2 &= a_2\, x_2 \\ \dot{x}_3 &= a_3\, x_3. \end{aligned}$$

These equations can evidently be solved separately to give

$$\begin{aligned} x_1 &= e^{a_1 t} c_1 \\ x_2 &= e^{a_2 t} c_2 \\ x_3 &= e^{a_3 t} c_3 \end{aligned}$$

for some constants c_1, c_2 and c_3. We can write the equations and the solution in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} e^{a_1 t} & 0 & 0 \\ 0 & e^{a_2 t} & 0 \\ 0 & 0 & e^{a_3 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

This was easy because the different variables do not interact with each other at all, so we just have diagonal matrices. If we call the diagonal matrix on the left D, then it is natural to call the matrix on the right e^{Dt}. With this notation, the equation is $\dot{x} = Dx$ and the solution is $x = e^{Dt}c$.

Now consider instead a system where we do have interaction, such as

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{aligned}$$

This can be written in matrix form as $\dot{x} = Ax$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

It is assumed here that the coefficients a_{ij} are constants, not depending on t or the variables x_i. Suppose we can diagonalise A, as $A = UDU^{-1}$ say, with

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

The equation $\dot{x} = Ax$ then becomes $\dot{x} = UDU^{-1}x$, or equivalently $U^{-1}\dot{x} = DU^{-1}x$. If we put $y = U^{-1}x$, this becomes $\dot{y} = Dy$. As D is diagonal we can solve for the three variables y_i separately, giving $y_i = e^{\lambda_i t}b_i$ for some constants b_i, or equivalently $y = e^{Dt}b$. This in turn gives $x = Uy = Ue^{Dt}b$. It is sometimes convenient to put $c = Ub$, so $b = U^{-1}c$, and the solution can be written as $x = Ue^{Dt}U^{-1}c$. When $t = 0$ we find that e^{Dt} is the identity matrix and so $x = UU^{-1}c = c$. Thus, c is just the initial value of x.

Example 15.1. Suppose that

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 + x_3 \\ \dot{x}_2 &= 2x_2 + 2x_3 \\ \dot{x}_3 &= 3x_3,\end{aligned}$$

with $x_1 = x_2 = 0$ and $x_3 = 1$ at $t = 0$. This can be written as $\dot{x} = Ax$, where A is the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ occurring in Example 14.6. As we saw there, we have a diagonalisation $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that $x = Ue^{Dt}U^{-1}c$, where $c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ is the specified initial value for x. This gives

$$\begin{aligned}
x &= \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^t & e^{2t} & \frac{3}{2}e^{3t} \\ 0 & e^{2t} & 2e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1/2 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}e^t - 2e^{2t} + \frac{3}{2}e^{3t} \\ -2e^{2t} + 2e^{3t} \\ e^{3t} \end{bmatrix}.
\end{aligned}$$

Example 15.2. Suppose we have

$$\dot{x} = \dot{y} = \dot{z} = x + y + z$$

with $x = z = 0$ and $y = 1$ at $t = 0$. This can be written as $\dot{v} = Av$, where

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}
\chi_A(t) &= \det \begin{bmatrix} 1-t & 1 & 1 \\ 1 & 1-t & 1 \\ 1 & 1 & 1-t \end{bmatrix} \\
&= (1-t)((1-t)^2 - 1) - (1-t-1) + (1-(1-t)) \\
&= 3t^2 - t^3 = t^2(3-t).
\end{aligned}$$

This shows that the eigenvalues are 0 and 3. If we put

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we find that $Au_1 = Au_2 = 0$ and $Au_3 = 3u_3$. Thus, the vectors u_i form a basis for $\mathbb{R}^3$ consisting of eigenvectors for A, with eigenvalues 0, 0 and 3 respectively. This means that we have a diagonalisation $A = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We can find U^{-1} by the following row-reduction:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & -1 & 0 & -1/3 & -1/3 & 2/3 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right].$$

The conclusion is that

$$U^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

The solution to our differential equation is now $v = Ue^{Dt}U^{-1}c$, where c is the initial value of v. We are given that $x = z = 0$

and $y = 1$ at $t = 0$, which means that $c = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} 1 & 0 & e^{3t} \\ -1 & 1 & e^{3t} \\ 0 & -1 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (e^{3t} - 1)/3 \\ (e^{3t} + 2)/3 \\ (e^{3t} - 1)/3 \end{bmatrix} .$$

16. Difference equations

We now discuss another kind of naturally occurring problem that can be solved by diagonalisation. It should be admitted that the method explained here is not the most efficient possible, but it explains some ideas that are less visible in more streamlined methods.

Example 16.1. Suppose it is given that the sequence $a_0, a_1, a_2, \ldots$ satisfies

(a) $a_0 = -1$
(b) $a_1 = 0$
(c) $a_{i+2} = 6a_{i+1} - 8a_i$ for all $i \geq 2$.

This gives

$$a_2 = 6a_1 - 8a_0 = 6 \times 0 - 8 \times (-1) = 8$$
$$a_3 = 6a_2 - 8a_1 = 6 \times 8 - 8 \times 0 = 48$$
$$a_4 = 6a_3 - 8a_2 = 6 \times 48 - 8 \times 8 = 224$$

and so on. We would like to find a formula giving a_n for all n. The first step is to reformulate the problem in terms of vectors and matrices. We put $v_i = \begin{bmatrix} a_i \\ a_{i+1} \end{bmatrix} \in \mathbb{R}^2$, so conditions (a) and (b) tell us that $v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Condition (c) tells us that

$$v_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ 6a_{n+1} - 8a_n \end{bmatrix} =$$
$$\begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix} v_n.$$

We write $A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix}$, so the above reads $v_{n+1} = Av_n$. In particular, we have

$$v_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$v_1 = Av_0 = A \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$v_2 = Av_1 = A^2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$v_3 = Av_2 = A^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and so on, so $v_n = A^n \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ for all n, which is already a useful answer. We can make it more explicit by diagonalising A. The characteristic polynomial is

$$\chi_A(t) = \det \begin{bmatrix} -t & 1 \\ -8 & 6-t \end{bmatrix} = t^2 - 6t + 8 = (t-2)(t-4),$$

so the eigenvectors are 2 and 4. Using the row-reductions

$$A - 2I = \begin{bmatrix} -2 & 1 \\ -8 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} -4 & 1 \\ -8 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}$$

we see that the vector $u_1 = \begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector of eigenvalue 2, and the vector $u_2 = \begin{bmatrix}1\\4\end{bmatrix}$ is an eigenvector of eigenvalue 4. We therefore have a diagonalisation $A = UDU^{-1}$, where $U = \begin{bmatrix}1 & 1\\2 & 4\end{bmatrix}$ (and therefore $U^{-1} = \begin{bmatrix}2 & -1/2\\-1 & 1/2\end{bmatrix}$) and $D = \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}$. This gives

$$\begin{aligned}
v_n &= A^n v_0 = UD^nU^{-1}v_0 \\
&= \begin{bmatrix}1 & 1\\2 & 4\end{bmatrix}\begin{bmatrix}2^n & 0\\0 & 4^n\end{bmatrix}\begin{bmatrix}2 & -1/2\\-1 & 1/2\end{bmatrix}\begin{bmatrix}-1\\0\end{bmatrix} \\
&= \begin{bmatrix}2^n & 4^n\\2^{n+1} & 4^{n+1}\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}4^n - 2^{n+1}\\4^{n+1} - 2^{n+2}\end{bmatrix}.
\end{aligned}$$

On the other hand, we have $v_n = \begin{bmatrix}a_n\\a_{n+1}\end{bmatrix}$ by definition, so

$$a_n = 4^n - 2^{n+1} = 2^{2n} - 2^{n+1}.$$

We will check that this formula does indeed give the required properties:

$$\begin{aligned}
a_0 &= 2^0 - 2^1 = 1 - 2 = -1 \\
a_1 &= 2^2 - 2^2 = 0 \\
6a_{i+1} - 8a_i &= 6(2^{2i+2} - 2^{i+2}) - 8(2^{2i} - 2^{i+1}) \\
&= 24 \times 2^{2i} - 24 \times 2^i - 8 \times 2^{2i} + 16 \times 2^i \\
&= 16 \times 2^{2i} - 8 \times 2^i = 2^{2i+4} - 2^{i+3} \\
&= 2^{2(i+2)} - 2^{(i+2)+1} = a_{i+2}.
\end{aligned}$$

Example 16.2. We will find the sequence satisfying $b_0 = 3$ and $b_1 = 6$ and $b_2 = 14$ and

$$b_{i+3} = 6b_i - 11b_{i+1} + 6b_{i+2}.$$

The first step is to note that the vectors $v_i = \begin{bmatrix} b_i & b_{i+1} & b_{i+2} \end{bmatrix}^T$ satisfy $v_0 = \begin{bmatrix} 3 & 6 & 14 \end{bmatrix}^T$ and

$$v_{i+1} = \begin{bmatrix} b_{i+1} \\ b_{i+2} \\ b_{i+3} \end{bmatrix} = \begin{bmatrix} b_{i+1} \\ b_{i+2} \\ 6b_i - 11b_{i+1} + 6b_{i+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \begin{bmatrix} b_i \\ b_{i+1} \\ b_{i+2} \end{bmatrix}$$

where $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$. It follows that $v_n = B^n v_0$ for all n, and b_n is the top entry in the vector v_n. To evaluate this more explicitly, we need to diagonalise B. The characteristic polynomial is

$$\chi_B(t) = \det(B - tI_3) = \det \begin{bmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 6 & -11 & 6-t \end{bmatrix}$$

$$= -t \det \begin{bmatrix} -t & 1 \\ -11 & 6-t \end{bmatrix} - \det \begin{bmatrix} 0 & 1 \\ 6 & 6-t \end{bmatrix} = -t(t^2 - 6t + 11)$$

$$= 6 - 11t + 6t^2 - t^3 = (1-t)(2-t)(3-t).$$

This shows that the eigenvalues are 1, 2 and 3. We can find the corresponding eigenvectors by row-reduction as in Method 13.14:

$$B - I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$B - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/9 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

This gives $B = UDU^{-1}$, where

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \qquad\qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We will use the adjugate method to calculate U^{-1}. (It would also work to row-reduce $[U|I_3]$ as in Method 11.11 instead.) The minor determinants are

$$\begin{aligned} m_{11} &= 6 & m_{12} &= 6 & m_{13} &= 2 \\ m_{21} &= 5 & m_{22} &= 8 & m_{23} &= 3 \\ m_{31} &= 1 & m_{32} &= 2 & m_{33} &= 1. \end{aligned}$$

This gives

$$\operatorname{adj}(U) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{bmatrix} = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}.$$

We also have

$$\det(U) = U_{11}m_{11} - U_{12}m_{12} + U_{13}m_{13} = 6 - 6 + 2 = 2,$$

so

$$U^{-1} = \frac{\operatorname{adj}(U)}{\det(U)} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix}.$$

This gives

$$\begin{aligned} v_n &= B^n v_0 = UD^nU^{-1}v_0 \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 14 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^n & 3^n \\ 1 & 2^{n+1} & 3^{n+1} \\ 1 & 2^{n+2} & 3^{n+2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + 2^n + 3^n \\ 1 + 2^{n+1} + 3^{n+1} \\ 1 + 2^{n+2} + 3^{n+2} \end{bmatrix}. \end{aligned}$$

Moreover, b_n is the top entry in v_n, so we conclude that
$$b_n = 1 + 2^n + 3^n.$$

Example 16.3. The Fibonacci numbers are defined by $F_0 = 0$ and $F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$. The vectors $v_i = \begin{bmatrix} F_i \\ F_{i+1} \end{bmatrix}$ therefore satisfy $v_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and
$$v_{n+1} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n + F_{n+1} \end{bmatrix} = Av_n,$$
where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. This has $\chi_A(t) = t^2 - t - 1$, which has roots $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$. To find an eigenvector of eigenvalue λ_1, we must solve
$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}.$$
The top row gives $y = \lambda_1 x$. The bottom row gives $x + y = \lambda_1 y$. After substituting $y = \lambda_1 x$ this becomes $x + \lambda_1 x = \lambda_1^2 x$, or $(\lambda_1^2 - \lambda_1 - 1)x = 0$. This holds automatically, because λ_1 is a root of the equation $t^2 - t - 1 = 0$. We can thus choose $x = 1$ to get an eigenvector $u_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ of eigenvalue λ_1. Similarly, we have an eigenvector $u_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$ of eigenvalue λ_2, giving a diagonalisation $A = UDU^{-1}$ with
$$U = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$
Here $\det(U) = \lambda_2 - \lambda_1 = -\sqrt{5}$ and so
$$U^{-1} = \frac{-1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}.$$

This gives

$$
\begin{aligned}
v_n &= A^n v_0 = UD^nU^{-1}v_0 \\
&= \frac{-1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^n & \lambda_2^n \\ \lambda_1^{n+1} & \lambda_2^{n+1} \end{bmatrix} \\
&= \begin{bmatrix} (\lambda_1^n - \lambda_2^n)/\sqrt{5} \\ (\lambda_1^{n+1} - \lambda_2^{n+1})/\sqrt{5} \end{bmatrix}.
\end{aligned}
$$

On the other hand, we have $v_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$ by definition, so

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

It is also useful to note here that $\lambda_1 \simeq 1.618033988$ and $\lambda_2 \simeq -0.6180339880$. As $|\lambda_1| > 1$ and $|\lambda_2| < 1$ we see that $|\lambda_1^n| \to \infty$ and $|\lambda_2^n| \to 0$ as $n \to \infty$. This means that when n is large we can neglect the λ_2 term and we have $F_n \simeq \lambda_1^n/\sqrt{5}$.

Remark 16.4. There are some strong and obvious common patterns in the above three examples, and we could streamline the method considerably by analysing and exploiting those patterns. However, we will not pursue that here.

17. Markov chains

Suppose we have a system X that can be in various different states, which we will label as state 1 to state n. Suppose that the system changes state in a random manner at regular intervals, say once per second. Suppose we know that X is in state 3 at time $t = 0$. How can we find the probability that X is in state 5 at $t = 26$?

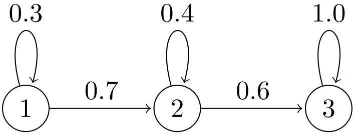
To study this question, we need to make a simplifying assumption. Suppose that X is in state i at time t. We will write $p_{j \leftarrow i}$ for the probability that X is in state j at time $t+1$. In principle this could depend on the time t, the earlier history at times $t-1$, $t-2$ and so on, or on various influences outside the system. We will assume that there is no such dependence: the transition probabilities $p_{j\leftarrow i}$ are just constants. A random system with these properties is called a *Markov chain.*

There are very many applications of the theory of Markov chains as models of random systems in economics, population biology, information technology and other areas.

Definition 17.1. For a Markov chain with n states, the *transition matrix* is the $n\times n$ matrix P with entry $p_{j\leftarrow i}$ in the i'th column of the j'th row. For example, when $n=4$ we have

$$P = \begin{bmatrix} p_{1\leftarrow 1} & p_{1\leftarrow 2} & p_{1\leftarrow 3} & p_{1\leftarrow 4} \\ p_{2\leftarrow 1} & p_{2\leftarrow 2} & p_{2\leftarrow 3} & p_{2\leftarrow 4} \\ p_{3\leftarrow 1} & p_{3\leftarrow 2} & p_{3\leftarrow 3} & p_{3\leftarrow 4} \\ p_{4\leftarrow 1} & p_{4\leftarrow 2} & p_{4\leftarrow 3} & p_{4\leftarrow 4} \end{bmatrix}.$$

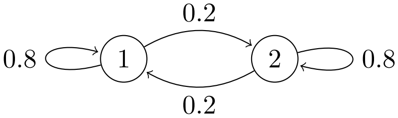
Example 17.2. We can draw a diagram of a 3-state Markov chain X as follows:



Each circle represents a state. The arrow labelled 0.7 (from state 1 to state 2) indicates that if X is in state 1, it will move to state 2 with probability 0.7. In other words $p_{2\leftarrow 1} = 0.7$. For the above diagram, the transition matrix is

$$P = \begin{bmatrix} p_{1\leftarrow 1} & p_{1\leftarrow 2} & p_{1\leftarrow 3} \\ p_{2\leftarrow 1} & p_{2\leftarrow 2} & p_{2\leftarrow 3} \\ p_{3\leftarrow 1} & p_{3\leftarrow 2} & p_{3\leftarrow 3} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.0 & 0.0 \\ 0.7 & 0.4 & 0.0 \\ 0.0 & 0.6 & 1.0 \end{bmatrix}.$$

Example 17.3. Consider a two-state Markov chain which stays in the same state with probability 0.8, and flips to the other state with probability 0.2. This can be drawn as follows:



Again, each circle represents a state. The arrow from state 1 to state 2 labelled 0.2 indicates that if X is in state 1, it will move to state 2 with probability 0.2. In other words $p_{2\leftarrow 1} = 0.2$.

The transition matrix is

$$P = \begin{bmatrix} p_{1\leftarrow 1} & p_{1\leftarrow 2} \\ p_{2\leftarrow 1} & p_{2\leftarrow 2} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}.$$

The transition matrix has an important property as follows:

Definition 17.4. A *probability vector* in $\mathbb{R}^n$ is a vector $q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}^T$ such that $0 \leq q_i \leq 1$ for all i, and $\sum_i q_i = 1$. A *stochastic matrix* is a matrix where every column is a probability vector.

Lemma 17.5. *The transition matrix of a Markov process is stochastic.*

Proof. Probabilities always lie between 0 and 1. If we are in state i at time t, then we must be in some state at time $t+1$ (and we cannot be in two different states) so the sum of the probabilites for the various different states must be equal to one. In other words, we have $\sum_j p_{j\leftarrow i} = 1$, and the probabilities $p_{j\leftarrow i}$ are the entries in the i'th column of P, so the column sums are equal to one as required. $\square$

Now suppose we know that at time t, the probability that X is in state i is q_i. It is clear that the vector $q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}^T$

is then a probability vector. Let q'_j be the probability that X is in state j at time $t+1$. By elementary probability theory, we have

$$q'_j = p_{j\leftarrow 1}q_1 + p_{j\leftarrow 2}q_2 + \cdots + p_{j\leftarrow n}q_n = \sum_i p_{j\leftarrow i}q_i.$$

This means that if we regard q and q' as vectors, we just have $q' = Pq$. For example, for a 3-state system, we have

$$q' = \begin{bmatrix} q'_1 \\ q'_2 \\ q'_3 \end{bmatrix} = \begin{bmatrix} p_{1\leftarrow 1}q_1 + p_{1\leftarrow 2}q_2 + p_{1\leftarrow 3}q_3 \\ p_{2\leftarrow 1}q_1 + p_{2\leftarrow 2}q_2 + p_{2\leftarrow 3}q_3 \\ p_{3\leftarrow 1}q_1 + p_{3\leftarrow 2}q_2 + p_{3\leftarrow 3}q_3 \end{bmatrix} =$$

$$\begin{bmatrix} p_{1\leftarrow 1} & p_{1\leftarrow 2} & p_{1\leftarrow 3} \\ p_{2\leftarrow 1} & p_{2\leftarrow 2} & p_{2\leftarrow 3} \\ p_{3\leftarrow 1} & p_{3\leftarrow 2} & p_{3\leftarrow 3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = Pq.$$

Definition 17.6. Let X be a Markov chain with n states. We write r_t for the vector in $\mathbb{R}^n$ whose i'th entry $(r_t)_i$ is the probability that X is in state i at time t. We call this vector the *distribution* for X at time t.

The argument outlined above proves the following result:

Proposition 17.7. *If P is the transition matrix for X, then $r_{t+1} = Pr_t$, and so $r_t = P^t r_0$ for all t.* □

Because of this, many questions about Markov chains reduce to understanding the powers of the matrix P, which we can do by diagonalising P.

Example 17.8. In Example 17.3 we considered a Markov chain with transition matrix $P = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$. The characteristic polynomial is $\chi_P(t) = t^2 - 1.6t + 0.6$ so the eigenvalues are $(1.6 \pm \sqrt{2.56 - 4 \times 0.6})/2$, which works out as 0.6 and 1. The vector

$u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of eigenvalue 0.6, and $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of eigenvalue 1. This gives a diagonalisation $P = UDU^{-1}$ with

$$U = [u_1|u_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix} \quad U^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix},$$

so

$$P^n = UD^nU^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (0.6)^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} =$$

$$\begin{bmatrix} 0.5(1+0.6^n) & 0.5(1-0.6^n) \\ 0.5(1-0.6^n) & 0.5(1-0.6^n) \end{bmatrix}$$

Suppose we are given that the system starts at $t = 0$ in state 1, so $r_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It follows that

$$r_n = P^n r_0 = \begin{bmatrix} 0.5(1+0.6^n) & 0.5(1-0.6^n) \\ 0.5(1-0.6^n) & 0.5(1-0.6^n) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5(1+0.6^n) \\ 0.5(1-0.6^n) \end{bmatrix}$$

Thus, at time n the probability of being in state 1 is $0.5(1 + 0.6^n)$, and the probability of being in state 2 is $0.5(1 - 0.6^n)$.

When n is large, we observe that $(0.6)^n$ will be very small, so $r_n \simeq \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, so it is almost equally probable that X will be in either of the two states. This should be intuitively plausible, given the symmetry of the situation.

Example 17.9. Consider the Markov chain X as in Example 17.2. Suppose that it starts in state 1 at time $t = 0$. What is the probability that it is in state 3 at time $t = 5$?

To calculate this, we need to know the eigenvalues and eigenvectors of the transtion matrix P. The characteristic polynomial is

$$\chi_P(t) = \det \begin{bmatrix} 0.3-t & 0.0 & 0.0 \\ 0.7 & 0.4-t & 0.0 \\ 0.0 & 0.6 & 1.0-t \end{bmatrix} = (0.3{-}t)(0.4{-}t)(1{-}t),$$

so the eigenvalues are 0.3, 0.4 and 1. To find an eigenvector of eigenvalue 0.3, we row-reduce the matrix $P - 0.3I$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 7/10 & 1/10 & 0 \\ 0 & 6/10 & 7/10 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1/7 & 0 \\ 0 & 1 & 7/6 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1/7 & 0 \\ 0 & 1 & 7/6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 7/6 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this we see that the eigenvectors of eigenvalue 0.3 are the vectors $\begin{bmatrix} x & y & z \end{bmatrix}^T$ satisfying $x - z/6 = 0$ and $y + 7z/6 = 0$, which means that $x = z/6$ and $y = -7z/6$ with z arbitrary. It will be convenient to take $z = 6$, giving $x = 1$ and $y = -7$, so we have an eigenvector $u_1 = \begin{bmatrix} 1 & -7 & 6 \end{bmatrix}^T$ of eigenvalue 0.3. By the same method we obtain an eigenvector $u_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ of eigenvalue 0.4, and an eigenvector $u_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ of eigenvalue 1. This gives a diagonalisation $P = UDU^{-1}$, where

$$U = [u_1|u_2|u_3] = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 6 & -1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}.$$

We can find U^{-1} by Method 11.11: from the row-reduction

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -7 & 1 & 0 & 0 & 1 & 0 \\ 6 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 7 & 1 & 0 \\ 0 & -1 & 1 & -6 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 7 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

we see that $U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. This gives

$$\begin{aligned} P^k &= UD^kU^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 6 & -1 & 1 \end{bmatrix} \begin{bmatrix} (0.3)^k & 0 & 0 \\ 0 & (0.4)^k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (0.3)^k & 0 & 0 \\ 7(0.4)^k - 7(0.3)^k & (0.4)^k & 0 \\ 1 + 6(0.3)^k - 7(0.4)^k & 1 - (0.4)^k & 1 \end{bmatrix}. \end{aligned}$$

Next, we are assuming that the system is definitely in state 1 at time $t = 0$, so the initial distribution is $r_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. It follows that

$$r_k = P^k r_0 = \begin{bmatrix} (0.3)^k & 0 & 0 \\ 7(0.4)^k - 7(0.3)^k & (0.4)^k & 0 \\ 1 + 6(0.3)^k - 7(0.4)^k & 1 - (0.4)^k & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} (0.3)^k \\ 7(0.4)^k - 7(0.3)^k \\ 1 + 6(0.3)^k - 7(0.4)^k \end{bmatrix}.$$

For the probability p that X is in state 3 at time $t = 5$, we need to take $k = 5$ and look at the third component, giving

$$p = 6(0.3)^5 - 7(0.4)^5 + 1 \simeq 0.94290.$$

In both of the last examples, one of the eigenvalues is equal to one. This is not a coincidence, as we now explain.

Proposition 17.10. *If P is an $n \times n$ stochastic matrix, then* 1 *is an eigenvalue of P.*

Proof. Let P be an $n \times n$ stochastic matrix, with columns $v_1, \dots, v_n$ say. Put $d = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix}^T \in \mathbb{R}^n$. Because P is stochastic we know that the sum of the entries in v_i is one, or in other words that $v_i.d = 1$. This means that

$$P^T d = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_n^T \end{array}\right] \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} v_1.d \\ \vdots \\ v_n.d \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = d.$$

Thus, d is an eigenvector of P^T with eigenvalue 1. It follows by Corollary 13.12 that 1 is also an eigenvalue of P, as required. □

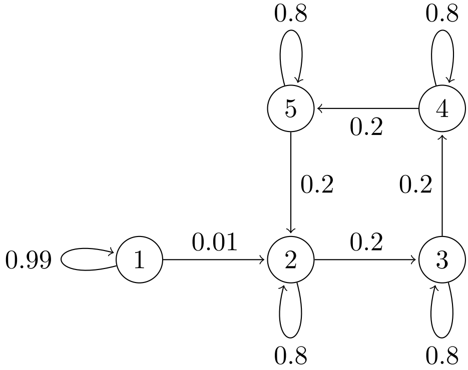
Definition 17.11. A *stationary distribution* for a Markov chain X is a probability vector q that satisfies $Pq = q$ (so q is an eigenvector of eigenvalue 1).

Remark 17.12. It often happens that the distribution vectors r_n converge (as $n \to \infty$) to a distribution r_∞, whose i'th component is the long term average probability of the system being in state i. Because $Pr_n = r_{n+1}$ we then have

$$Pr_\infty = P \lim_{n\to\infty} r_n = \lim_{n\to\infty} Pr_n = \lim_{n\to\infty} r_{n+1} = r_\infty,$$

so r_∞ is a stationary distribution. Moreover, it often happens that there is only one stationary distribution. There are many theorems about this sort of thing, but we will not explore them in this course.

Example 17.13. Consider the following Markov chain:



We will use a heuristic argument to guess what the stationary distribution should be, and then give a rigorous proof that it is correct.

At each time there is a (small but) nonzero probability of leaving state 1 and entering the square, so if we wait long enough we can expect this to happen. After we have entered the square there is no way we can ever return to state 1, so the long-run average probability of being in state 1 is zero. Once we have entered the square there is nothing to distinguish between the states 2, 3, 4 and 5, so by symmetry we expect to spend a quarter of the time in each of these states. This suggests that the vector $q = \begin{bmatrix} 0 & 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}^T$ should be a stationary distribution. Indeed, it is clear that q is a probability vector, so we just need to show that $Pq = q$, where

$$P = \begin{bmatrix} 0.99 & 0 & 0 & 0 & 0 \\ 0.01 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{bmatrix}$$

is the transition matrix. We have

$$Pq = \begin{bmatrix} 0.99 & 0 & 0 & 0 & 0 \\ 0.01 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} = q$$

as required.

18. PageRank

We next discuss the Google PageRank algorithm, which is an application of the theory of Markov chains to the problem of web search. Suppose we have a collection of n web pages with various links between them, and we let S_i denote the i'th page. The problem is to calculate a system of rankings for the web pages, which can be used to decide which should be returned first in response to a search query. The method that we will describe was a major advance when it was first introduced in 1996. Although much more complex methods are used now, the key insight behind PageRank is still important. The original paper describing PageRank and many other features of the original Google search system are described in this paper: `http://infolab.stanford.edu/pub/papers/google.pdf`.

The basic idea is that a link from S_j to S_i suggests that the author of S_j thought that S_i was important and reliable, so it should increase the rank of S_i. However, we should only take this seriously if we think that S_j itself is important and reliable, which we can assess by looking for links from other pages S_k to S_j, and so on. Moreover, if S_j contains a large number of links then that suggests that the author did not think that each link was individually very important, so that should also decrease the ranking effect of each link.

For a different perspective, we could imagine surfing the web randomly. Once per second, we choose one of the links on the current page at random, and click it. Let r_i denote the long run average proportion of the time that we spend on page i. It turns out that this is a reasonable measure of the quality of S_i, in terms of the principles mentioned above. Indeed, if there are lots of links from other pages S_j to S_i, then the probability of visiting S_i will clearly increase. However, this effect will only come into play if we are visiting S_j, so the strength of the effect is moderated by the probability of being at S_j, or in other words the rank r_j. Moreover, the effect will be weakened if there are many other links on S_j, because that will decrease the probability that we click the one that leads to S_i.

We next construct some equations to encode the informal discussion above. For simplicity we will assume that

(a) Each page S_j contains at least one link to another page in the collection.
(b) No page contains more than one link to the same target.

(These restrictions could be removed with minor modifications to the algorithm.) We let N_j be the total number of links from S_j. Assumption (a) says that $N_j > 0$, so it is meaningful to consider $1/N_j$. Assumption (b) means that N_j is also the number of pages to which S_j has a link. We define an $n \times n$ matrix P by the rule

$$P_{ij} = \begin{cases} 1/N_j & \text{if there is a link from } S_j \text{ to } S_i \\ 0 & \text{otherwise.} \end{cases}$$

Our random surfer model can be considered as an n-state Markov chain, with the state corresponding to the page that we are currently visiting. The probability of moving from state j to state i is then the number P_{ij} as above. In other words,

P is the transition matrix for this Markov chain. Note that P is indeed a stochastic matrix, as it should be: in column j we have N_j nonzero entries all equal to $1/N_j$, so the sum is one.

Now let r_i denote the ranking for the page S_i. It is natural to normalise these rankings by requiring that $r_i \geq 0$ for all i and $\sum_i r_i = 1$. This means that the vector $r \in \mathbb{R}^n$ is a probability vector. Next, we construct an equation that represents our ranking principle as discussed previously. Whenever there is a link from S_j to S_i, this should give a contribution to r_i. The strength of this should be an increasing function of the ranking r_j of the page that carries the link, and a decreasing function of the number N_j of links on the page. It is therefore natural to take the contribution to be r_j/N_j. On the other hand, for pages S_j that do not link to S_i, we get a contribution of zero. This means that we should have

$$r_i = \sum_j P_{ij} r_j,$$

or equivalently $r = Pr$, so r is an eigenvector of P with eigenvalue 1. As it is also a probability vector, we see that it is a stationary distribution for our Markov chain. As we have mentioned previously, most but not all Markov chains have a unique stationary distribution, so it is reasonable to hope that there is a unique system of rankings satisfying the above equations.

One way to calculate r would be to solve the equations $\sum_i r_i = 1$ and $r_i = \sum_j P_{ij} r_j$ by row-reduction. This is no problem when the number n of web pages is small and so we have small matrices to row-reduce. However, it is not feasible when indexing millions or billions of documents. However, we can use another method based on Remark 17.12 instead. Let $q \in \mathbb{R}^n$ be the probability vector with $q_i = 1/n$ for all i, which represents a situation in which we have chosen one of our n

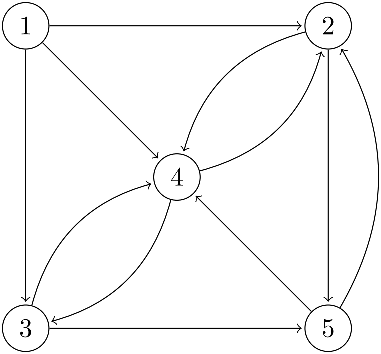
pages completely at random. In typical cases, the vectors $P^k q$ will converge quite quickly to the unique stationary distribution r. Thus, we can find an approximation to r by calculating $P^k q$ for some fairly small value of k. If the calculation is organised in an intelligent way, this is feasible even when n is very large.

We next discuss a slight modification to the above scheme, which was found by Google to improve the rankings. Fix a number d with $0 < d < 1$, called the *damping factor* (a value of 0.85 was found to work well). We now imagine surfing the web randomly as before, except that with probability d we click a randomly chosen link on the current page, and with probability $1 - d$ we just choose a page completely at random, whether or not there is a link. The new transition probabilities are

$$Q_{ij} = \begin{cases} \frac{d}{N_j} + \frac{1-d}{n} & \text{if there is a link from } S_j \text{ to } S_i \\ \frac{1-d}{n} & \text{otherwise.} \end{cases}$$

Another way to write this is to let R be the stochastic matrix with $R_{ij} = 1/n$ for all i and j; then $Q = dP + (1-d)R$. There is then a unique probability vector r' with $Qr' = r'$, and the entries in r' can be used to rank the pages S_i.

As an example, consider the following web of pages and links:



The corresponding matrix P is as follows:

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 1/2 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{bmatrix}.$$

We can use Maple to calculate the corresponding ranking vector:

```
with(LinearAlgebra):
n := 5;

P := <<  0 |   0 | 0   |   0 |   0 >,
      <1/3 |   0 | 0   | 1/2 | 1/2 >,
      <1/3 |   0 | 0   | 1/2 |   0 >,
      <1/3 | 1/2 | 1/2 |   0 | 1/2 >,
      <  0 | 1/2 | 1/2 |   0 |   0 >>;

NS := NullSpace(P - IdentityMatrix(n));
r := NS[1];
r := r / add(r[i],i=1..n);
```

```
r := evalf(r);
```

The first two commands define n and P. Next, we want to find r, which must satisfy $Pr = r$ or in other words $(P - I_n)r = 0$. The command `NS := NullSpace(P - IdentityMatrix(n))` defines `NS` to be a set of vectors that satisfy this equation. In this case (as in most cases) this is a set with only one element. The command `r := NS[1]` defines r to be the first (and only) element in the set `NS`. We now have $Pr = r$, but the normalisation condition $\sum_i r_i = 1$ need not be satisfied, or equivalently, r need not be a probability vector. This will be fixed if we just divide r by $\sum_i r_i$, which is the effect of the command `r := r / add(r[i],i=1..n);`. Finally, it is convenient to rewrite all fractions as (approximate) decimals, which we do using the `evalf()` function. At the end, Maple gives the result

$$r = \begin{bmatrix} 0.0 \\ 0.2777777778 \\ 0.1666666667 \\ 0.3333333333 \\ 0.2222222222 \end{bmatrix}.$$

This means that page 1 has rank 0.0 (because there are no links to it), page 2 has rank 0.277 and so on. We can also use Maple to verify that r is the limit of the vectors $P^k q$:

```
q := Vector(n,[1/n $ n]);
evalf(P^10 . q);
```

The first line means: we define q to be a vector of length n, whose entries are $1/n$ repeated n times. Thus, it is just a Maple translation of our previous definition of q. The next line tells us that

$$P^{10}q = \begin{bmatrix} 0.0 \\ 0.2783203125 \\ 0.1667317708 \\ 0.3332682292 \\ 0.2216796875 \end{bmatrix},$$

which is fairly close to r.

We next modify the calculation to include damping:

```
d := 0.85;
R := Matrix(n,n,[1/n $ n^2]);
Q := d * P + (1-d) * R;
NS := NullSpace(Q - IdentityMatrix(n));
r := NS[1];
r := r / add(r[i],i=1..n);
r := evalf(r);
```

The first line sets the damping factor d. The second line means: we define R to be a matrix of size $n \times n$ whose entries are $1/n$ repeated n^2 times. Thus, it is just a Maple translation of our previous definition of R. We then put $Q = dP + (1-d)R$, and the remaining lines are as for the undamped case. The conclusion is that

$$r = \begin{bmatrix} 0.030000000000000 \\ 0.264738919667590 \\ 0.172934210526316 \\ 0.316315789473684 \\ 0.216011080332410 \end{bmatrix}.$$

The rankings are different from those produced by the undamped method, but not by much. In particular, the page S_1 has a small positive rank, even though there are no links to it.

19. Subspaces of $\mathbb{R}^n$

When considering the geometry of two or three-dimensional space, we often consider lines and planes, and the ones that pass through the origin form an important special case. In this section we will start to discuss the analogous structures in $\mathbb{R}^n$, where n may be bigger than 3.

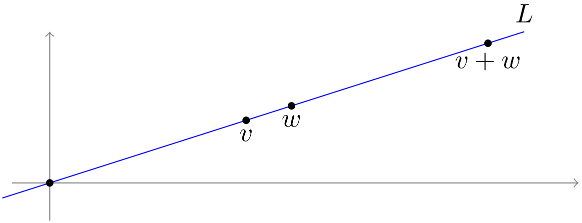
Definition 19.1. A subset $V \subseteq \mathbb{R}^n$ is a *subspace* if

(a) The zero vector is an element of V.
(b) Whenever v and w are two elements of V, the sum $v + w$ is also an element of V. (In other words, V is closed under addition.)
(c) Whenever v is an element of V and t is a real number, the vector tv is again an element of V. (In other words, V is closed under scalar multiplication.)

Remark 19.2. The same word "subspace" is sometimes used for various different concepts. If we need to emphasise that we are using the above definition, we may say "vector subspace" or "linear subspace" rather than just "subspace".

Example 19.3. Let L be the line in $\mathbb{R}^2$ with equation $y = x/\pi$. We claim that this is a subspace. Indeed:

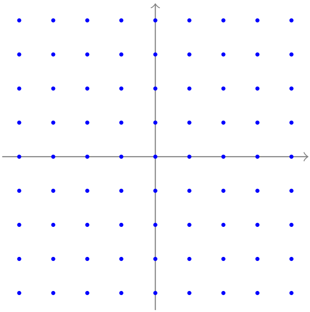
- The point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ lies on L, so condition (a) is satisfied.
- Suppose we have two points $v, w \in L$, so $v = \begin{bmatrix} a \\ a/\pi \end{bmatrix}$ and $w = \begin{bmatrix} b \\ b/\pi \end{bmatrix}$ for some numbers a and b. Then $v + w = \begin{bmatrix} a+b \\ (a+b)/\pi \end{bmatrix}$, which again lies on L. Thus, condition (b) is satisfied.



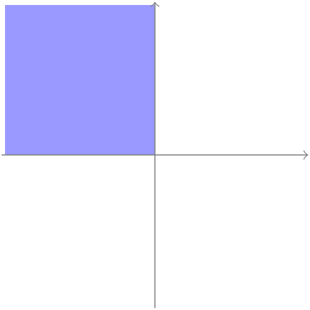
- Suppose again that $v \in L$, so $v = \begin{bmatrix} a \\ a/\pi \end{bmatrix}$ for some a. Suppose also that $t \in \mathbb{R}$. Then $tv = \begin{bmatrix} ta \\ ta/\pi \end{bmatrix}$, which again lies on L, so condition (c) is satisfied.

Example 19.4. Consider the following subsets of $\mathbb{R}^2$:

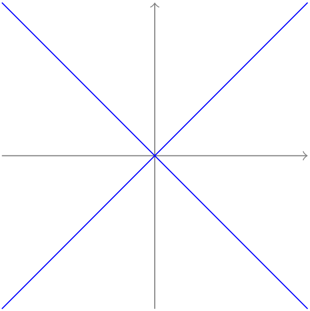
$$
\begin{aligned}
V_1 &= \mathbb{Z}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \text{ and } y \text{ are integers } \right\} \\
V_2 &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \leq 0 \leq y \right\} \\
V_3 &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x^2 = y^2 \right\} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = \pm y \right\}.
\end{aligned}
$$



V_1



V_2



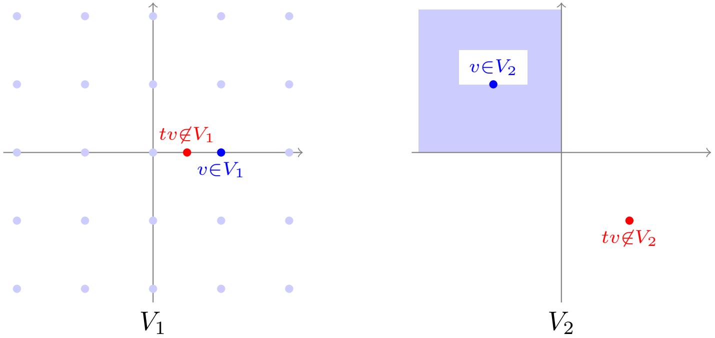
V_3

We claim that none of these are subspaces. We first consider V_1. It is clear that the zero vector has integer coordinates and so lies in V_1, so condition (a) is satisfied. Next, if v and w both have integer coordinates then so does $v + w$. In other words, if $v, w \in V_1$ then also $v + w \in V_1$. This shows that condition (b) is also satisfied. However, condition (c) is not satisfied. Indeed, if we take $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $t = 0.5$ then $v \in V_1$

and $t \in \mathbb{R}$ but the vector $tv = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$ does not lie in V_1. (This is generally the best way to prove that a set is not a subspace: provide a completely specific and explicit example where one of the conditions is not satisfied.)

We now consider V_2. As $0 \leq 0 \leq 0$ we see that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V_2$, so condition (a) is satisfied. Now suppose we have two vectors $v, v' \in V_2$, so $v = \begin{bmatrix} x \\ y \end{bmatrix}$ and $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ with $x \leq 0 \leq y$ and $x' \leq 0 \leq y'$. As $x, x' \leq 0$ it follows that $x + x' \leq 0$. As $y, y' \geq 0$ it follows that $y + y' \geq 0$. This means that the sum $v + v' = \begin{bmatrix} x + x' \\ y + y' \end{bmatrix}$ is again in V_2, so condition (b) is satisfied. However, condition (c) is again not satisfied. Indeed, if we take $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $t = -1$ then $v \in V_2$ and $t \in \mathbb{R}$ but the vector $tv = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ does not lie in V_2.

Finally, we consider V_3. It is again clear that condition (a) is satisfied. Now suppose we have $v = \begin{bmatrix} x \\ y \end{bmatrix} \in V_3$ (so $x^2 = y^2$) and $t \in \mathbb{R}$. It follows that $(tx)^2 = t^2x^2 = t^2y^2 = (ty)^2$, so the vector $tv = \begin{bmatrix} tx \\ ty \end{bmatrix}$ again lies in V_3. This means that condition (c) is satisfied. However, condition (b) is not satisfied, because the vectors $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ lie in V_3, but $v + w$ does not.



Example 19.5. For any n there are two extreme examples of subspaces of $\mathbb{R}^n$. Firstly, we have the set $\{0\}$ consisting of just the zero vector. As $0 + 0 = 0$ and $t0 = 0$ for all $t \in \mathbb{R}$, we see that $\{0\}$ is closed under addition and scalar multiplication, so it is a subspace. Similarly, the full set $\mathbb{R}^n$ is also closed under addition and scalar multiplication, so it is a subspace of itself.

Proposition 19.6. *Let V be a subspace of $\mathbb{R}^n$. Then any linear combination of elements of V is again in V.*

Proof. Suppose we have elements $v_1, \ldots, v_k \in V$, and suppose that w is a linear combination of the v_i, say $w = \sum_i \lambda_i v_i$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. As $v_i \in V$ and $\lambda_i \in \mathbb{R}$, condition (c) tells us that $\lambda_i v_i \in V$. Now $\lambda_1 v_1$ and $\lambda_2 v_2$ are elements of V, so condition (b) tells us that $\lambda_1 v_1 + \lambda_2 v_2 \in V$. Next, as $\lambda_1 v_1 + \lambda_2 v_2 \in V$ and $\lambda_3 v_3 \in V$ condition (b) tells us that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \in V$. By extending this in the obvious way, we eventually conclude that the vector $w = \lambda_1 v_1 + \cdots + \lambda_k v_k$ lies in V as claimed.

Condition (a) can also be thought of as saying that the sum of no terms lies in V; this is a kind of degenerate linear combination. $\square$

Proposition 19.7. *Let V be a subspace of $\mathbb{R}^2$. Then V is either $\{0\}$ or all of $\mathbb{R}^2$ or a straight line through the origin.*

Proof. (a) If $V = \{0\}$ then there is nothing more to say.

(b) Suppose that V contains two vectors v and w such that the list (v, w) is linearly independent. As this is a linearly independent list of two vectors in $\mathbb{R}^2$, Proposition 10.12 tells us that it must be a basis. Thus, every vector $x \in \mathbb{R}^2$ is a linear combination of v and w, and therefore lies in V by Proposition 19.6. Thus, we have $V = \mathbb{R}^2$.

(c) Suppose instead that neither (a) nor (b) holds. As (a) does not hold, we can choose a nonzero vector $v \in V$. Let L be the set of all scalar multiples of v, which is a straight line through the origin. As V is a subspace and $v \in V$ we know that every multiple of v lies in V, or in other words that $L \subseteq V$. Now let w be any vector in V. As (b) does not hold, the list (v, w) is linearly dependent, so Lemma 8.5 tells us that w is a multiple of v and so lies in L. This shows that $V \subseteq L$, so $V = L$.

□

Definition 19.8. Let $\mathcal{W} = (w_1, \ldots, w_r)$ be a list of vectors in $\mathbb{R}^n$.

(a) $\mathrm{span}(\mathcal{W})$ is the set of all vectors $v \in \mathbb{R}^n$ that can be expressed as a linear combination of the list $\mathcal{W}$.

(b) $\mathrm{ann}(\mathcal{W})$ is the set of all vectors $u \in \mathbb{R}^n$ such that $u.w_1 = \cdots = u.w_r = 0$.

We will show in Propositions 19.23 and 19.24 that $\mathrm{span}(\mathcal{W})$ and $\mathrm{ann}(\mathcal{W})$ are both subspaces of $\mathbb{R}^n$. First, however, we give some examples to illustrate the definitions.

Remark 19.9. The terminology in (a) is related in an obvious way to the terminology in Definition 9.1: the list $\mathcal{W}$ spans $\mathbb{R}^n$ if and only if every vector in $\mathbb{R}^n$ is a linear combination of $\mathcal{W}$, or in other words $\operatorname{span}(\mathcal{W}) = \mathbb{R}^n$.

If a subspace is given by a system of homogeneous equations, then it can easily be described as an annihilator, as in the following examples.

Example 19.10. Let V be the set of vectors $a = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ in $\mathbb{R}^4$ that satisfy $w + 2x = 2y + 3z$ and $w - z = x - y$. These equations can be rewritten as $w + 2x - 2y - 3z = 0$ and $w - x + y - z = 0$, or equivalently as

$$a.\begin{bmatrix} 1 & 2 & -2 & -3 \end{bmatrix}^T = a.\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T = 0.$$

Thus, we have

$$V = \operatorname{ann}(\begin{bmatrix} 1 & 2 & -2 & -3 \end{bmatrix}^T, \quad \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T).$$

Example 19.11. Similarly, put

$$W = \left\{ v = \begin{bmatrix} p \\ q \\ r \\ s \\ t \end{bmatrix} \in \mathbb{R}^5 \mid \begin{array}{rl} p+q+r & = 0, \\ q+r+s & = 0 \text{ and} \\ r+s+t & = 0 \end{array} \right\}$$

If we put

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

then the defining equations $p+q+r = q+r+s = r+s+t = 0$ can be rewritten as $a_1.v = a_2.v = a_3.v = 0$. This means that $W = \operatorname{ann}(a_1, a_2, a_3)$.

Remark 19.12. It is important here that the equations are homogeneous. Suppose instead we consider the set of solutions of a system of inhomogeneous equations, involving some pure constants as well as constants multiplied by variables. For example, we could consider

$$V = \{\begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \mid x + 2y = 4, \quad 3y + 4z = 5\}.$$

The vector $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ does not satisfy these equations and so is not an element of V. Thus, V is not a subspace of $\mathbb{R}^3$, and cannot be described as an annihilator.

We next discuss another situation in which annihilators appear in slightly disguised form.

Definition 19.13. Let A be an $m \times n$ matrix (so for each vector $v \in \mathbb{R}^n$ we have a vector $Av \in \mathbb{R}^m$). We put

$$\ker(A) = \{v \in \mathbb{R}^n \mid Av = 0\},$$

and call this the *kernel* of V.

Proposition 19.14. *The kernel of A is the annihilator of the transposed rows of A. In more detail, if*

$$A = \left[\begin{array}{c} u_1^T \\ \hline \vdots \\ \hline u_m^T \end{array}\right].$$

then $\ker(A) = \operatorname{ann}(u_1, \ldots, u_m)$.

Proof. We observed in Section 3 that

$$Av = \begin{bmatrix} u_1.v & \cdots & u_m.v \end{bmatrix}^T.$$

Thus, v lies in $\ker(A)$ iff $Av = 0$ iff $u_1.v = \cdots = u_m.v = 0$ iff v lies in $\operatorname{ann}(u_1, \ldots, u_m)$. $\square$

Example 19.15. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 30 & 20 & 10 \\ 100 & 100 & 100 \end{bmatrix},$$

so if $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T$ we have

$$Av = \begin{bmatrix} x + 2y + 3z \\ 30x + 20y + 10z \\ 100x + 100y + 100z \end{bmatrix}.$$

We thus have

$$\begin{aligned}
\ker(A) = \{v \mid Av = 0\} &= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \begin{bmatrix} x + 2y + 3z \\ 30x + 20y + 10z \\ 100x + 100y + 100z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \\
&= \left\{ v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid v.\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = v.\begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix} = v.\begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} = 0 \right\} \\
&= \operatorname{ann}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}, \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} \right)
\end{aligned}$$

Spans also commonly appear in slight disguise. The following examples are typical:

Example 19.16. Let V be the set of all vectors of the form

$$v = \begin{bmatrix} 2p + 5q + 3r \\ 9p - 4q + 2r \\ 3p + 3q + 3r \\ 8p - 4q - 5r \end{bmatrix}$$

for some $p, q, r \in \mathbb{R}$ (so V is a subset of $\mathbb{R}^4$). We can rewrite this as

$$v = p \begin{bmatrix} 2 \\ 9 \\ 3 \\ 8 \end{bmatrix} + q \begin{bmatrix} 5 \\ -4 \\ 3 \\ -4 \end{bmatrix} + r \begin{bmatrix} 3 \\ 2 \\ 3 \\ -5 \end{bmatrix}.$$

Thus, if we put

$$a = \begin{bmatrix} 2 \\ 9 \\ 3 \\ 8 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ -4 \\ 3 \\ -4 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 2 \\ 3 \\ -5 \end{bmatrix},$$

then the general form for elements of V is $v = pa + qb + rc$. In other words, the elements of V are precisely the vectors that can be expressed as a linear combination of a, b and c. This means that $V = \operatorname{span}(a, b, c)$.

Example 19.17. Let W be the set of all vectors in $\mathbb{R}^5$ of the form

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} t_3 - t_2 \\ t_1 + t_3 \\ t_2 - t_1 \\ t_1 + t_2 \\ t_3 + t_2 \end{bmatrix}$$

for some $t_1, t_2, t_3 \in \mathbb{R}$. This general form can be rewritten as

$$w = t_1 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = t_1 a_1 + t_2 a_2 + t_3 a_3,$$

where

$$a_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, we see that a vector $w \in \mathbb{R}^5$ lies in W iff it can be expressed as a linear combination of the vectors a_1, a_2 and a_3. This means that $W = \operatorname{span}(a_1, a_2, a_3)$.

Definition 19.18. Let A be an $m \times n$ matrix (so for each vector $t \in \mathbb{R}^n$ we have a vector $At \in \mathbb{R}^m$). We write $\operatorname{img}(A)$ for the set of vectors $w \in \mathbb{R}^m$ that can be expressed in the form $w = At$ for some $t \in \mathbb{R}^n$. We call this the *image* of the matrix A.

Proposition 19.19. *The image of A is just the span of the columns of A. In other words, if*

$$A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array}\right]$$

then $\operatorname{img}(A) = \operatorname{span}(v_1, \dots, v_n)$.

Proof. Recall from Section 3 that

$$At = \left[\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array}\right] \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} = t_1 v_1 + \cdots + t_n v_n.$$

Note here that each t_i is a scalar (the i'th entry in the vector t) whereas v_i is a vector (the i'th column in the matrix A). Thus At is the sum of the arbitrary scalars t_i multiplied by the given vectors v_i; in other words, it is an arbitrary linear combination of $v_1, \dots, v_n$. The claim is clear from this. $\square$

Example 19.20. Put

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 9 & -4 & 2 \\ 3 & 3 & 3 \\ 8 & -4 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then the space V in Example 19.16 is the image of A, and the space W in Example 19.17 is the image of B.

It turns out that the span of any list of vectors can also be described as the image of some other list of vectors. Conversely, the annihilator of any list of vectors can also be described as the span of a different list. In this section we will just give some examples; later we will discuss a general method to perform this kind of conversion.

Example 19.21. Consider the plane P in $\mathbb{R}^3$ with equation $x + y + z = 0$. More formally, we have

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x + y + z = 0 \right\}.$$

If we put $v = \begin{bmatrix} x & y & z \end{bmatrix}^T$ and $t = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, then we have $v.t = x + y + z$. It follows that

$$P = \{v \in \mathbb{R}^3 \mid v.t = 0\} = \operatorname{ann}(t).$$

On the other hand, if $x + y + z = 0$ then $z = -x - y$ so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x - y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Thus, if we put $u_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ then

$$P = \{x\,u_1 + y\,u_2 \mid x, y \in \mathbb{R}\} = \{\text{ linear combinations of } u_1 \text{ and } u_2\}$$

Example 19.22. Put

$$V = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w+2x+3y+4z = 4w+3x+2y+z = 0\}$$

If we put $a = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$ and $b = \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}^T$ then

$$w+2x+3y+4z = a.\begin{bmatrix} w & x & y & z \end{bmatrix}^T \qquad 4w+3x+2y+z = b.\big[w$$

so we can describe V as $\mathrm{ann}(a, b)$. On the other hand, suppose we have a vector $v = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ in V, so that

$$w + 2x + 3y + 4z = 0 \tag{A}$$

$$4w + 3x + 2y + z = 0 \tag{B}$$

If we subtract 4 times (A) from (B) and then divide by -15 we get equation (C) below. Similarly, if we subtract 4 times (B) from (A) and divide by -15 we get (D).

$$\tfrac{1}{3}x + \tfrac{2}{3}y + z = 0 \tag{C}$$

$$w + \tfrac{2}{3}x + \tfrac{1}{3}y = 0 \tag{D}$$

It follows that

$$v = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x - \frac{1}{3}y \\ x \\ y \\ -\frac{1}{3}x - \frac{2}{3}y \end{bmatrix} = x \begin{bmatrix} -2/3 \\ 1 \\ 0 \\ -1/3 \end{bmatrix} + y \begin{bmatrix} -1/3 \\ 0 \\ 1 \\ -2/3 \end{bmatrix}.$$

Using this we see that V can also be described as $\mathrm{span}(c, d)$, where

$$c = \begin{bmatrix} -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \end{bmatrix}^T \qquad d = \begin{bmatrix} -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \end{bmatrix}^T.$$

Proposition 19.23. *For any list $\mathcal{W} = (w_1, \dots, w_r)$ of vectors in $\mathbb{R}^n$, the set $\mathrm{ann}(\mathcal{W})$ is a subspace of $\mathbb{R}^n$.*

Proof. (a) The zero vector clearly has $0.w_i = 0$ for all i, so $0 \in \mathrm{ann}(\mathcal{W})$.

(b) Suppose that $u, v \in \mathrm{ann}(\mathcal{W})$. This means that $u.w_i = 0$ for all i, and that $v.w_i = 0$ for all i. It follows that $(u+v).w_i = u.w_i + v.w_i = 0 + 0 = 0$ for all i, so $u + v \in \mathrm{ann}(\mathcal{W})$. Thus, $\mathrm{ann}(\mathcal{W})$ is closed under addition.

(c) Suppose instead that $u \in \mathrm{ann}(\mathcal{W})$ and $t \in \mathbb{R}$. As before, we have $u.w_i = 0$ for all i. It follows that $(tu).w_i = t(u.w_i) = 0$ for all i, so $tu \in \mathrm{ann}(\mathcal{W})$. Thus, $\mathrm{ann}(\mathcal{W})$ is closed under scalar multiplication.

□

Proposition 19.24. *For any list $\mathcal{W} = (w_1, \ldots, w_r)$ of vectors in $\mathbb{R}^n$, the set* $\mathrm{span}(\mathcal{W})$ *is a subspace of* $\mathbb{R}^n$.

Proof. (a) The zero vector can be written as a linear combination $0 = 0w_1 + \cdots + 0w_r$, so $0 \in \mathrm{span}(\mathcal{W})$.

(b) Suppose that $u, v \in \mathrm{span}(\mathcal{W})$. This means that for some sequence of coefficients $\lambda_i \in \mathbb{R}$ we have $u = \sum_i \lambda_i w_i$, and for some sequence of coefficients μ_i we have $v = \sum_i \mu_i w_i$. If we put $\nu_i = \lambda_i + \mu_i$ we then have $u + v = \sum_i \nu_i w_i$. This expresses $u + v$ as a linear combination of $\mathcal{W}$, so $u + v \in \mathrm{span}(\mathcal{W})$. Thus, $\mathrm{span}(\mathcal{W})$ is closed under addition.

(c) Suppose instead that $u \in \mathrm{span}(\mathcal{W})$ and $t \in \mathbb{R}$. As before, we have $u = \sum_i \lambda_i w_i$ for some sequence of coefficients λ_i. If we put $\kappa_i = t\lambda_i$ we find that $tu = \sum_i \kappa_i w_i$, which expresses tu as a linear combination of $\mathcal{W}$, so $tu \in \mathrm{span}(\mathcal{W})$. Thus, $\mathrm{span}(\mathcal{W})$ is closed under scalar multiplication.

□

20. Bases for subspaces

We can now generalise the notion of a basis, as follows.

Definition 20.1. Let V be a subspace of $\mathbb{R}^n$. A *basis* for V is a linearly independent list $\mathcal{V} = (v_1, \ldots, v_r)$ of vectors in $\mathbb{R}^n$ such that $\mathrm{span}(\mathcal{V}) = V$.

This definition leads us to think about the possibilities for linearly independent lists in V. The empty list always counts as being linearly independent, and if $V = \{0\}$ that will be the only possibility. If $V \neq \{0\}$ then we can choose a nonzero vector $v_1 \in V$, and the list (v_1) will be linearly independent. There might or might not exist a linearly independent list of length two or more. However, any linearly independent list in V is in particular a linearly independent list in $\mathbb{R}^n$, so it has length at most n by Remark 8.9.

Definition 20.2. Let V be a subspace of $\mathbb{R}^n$. The *dimension* of V (written $\dim(V)$) is the maximum possible length of any linearly independent list in V (so $0 \leq \dim(V) \leq n$).

Proposition 20.3. *Let V be a subspace of $\mathbb{R}^n$, and put $d = \dim(V)$. Then any linearly independent list of length d in V is automatically a basis. In particular, V has a basis.*

Proof. Let $\mathcal{V} = (v_1, \ldots, v_d)$ be a linearly independent list of length d in V. Let u be an arbitrary vector in V. Consider the list $\mathcal{V}' = (v_1, \ldots, v_d, u)$. This is a list in V of length $d+1$, but d is the maximum possible length for any linearly independent list in V, so the list $\mathcal{V}'$ must be dependent. This means that there is a nontrivial relation

$$\lambda_1 v_1 + \cdots + \lambda_d v_d + \mu u = 0.$$

We claim that μ cannot be zero. Indeed, if $\mu = 0$ then the relation would become $\lambda_1 v_1 + \cdots + \lambda_d v_d = 0$, but $\mathcal{V}$ is linearly independent so this would give $\lambda_1 = \cdots = \lambda_d = 0$ as well as $\mu = 0$, so the original relation would be trivial, contrary to our assumption. Thus $\mu \neq 0$, so the relation can be rearranged as

$$u = -\frac{\lambda_1}{\mu} v_1 - \cdots - \frac{\lambda_d}{\mu} v_d,$$

which expresses u as a linear combination of $\mathcal{V}$. This shows that an arbitrary vector $u \in V$ can be expressed as a linear combination of $\mathcal{V}$, or in other words $V = \operatorname{span}(\mathcal{V})$. As $\mathcal{V}$ is also linearly independent, it is a basis for V. □

Once we have chosen a basis for a subspace $V \subseteq \mathbb{R}^n$ we can use it to identify V with $\mathbb{R}^r$ for some r. In more detail:

Proposition 20.4. *Let V be a subspace of $\mathbb{R}^n$, and let $\mathcal{V} = (v_1, \ldots, v_r)$ be a basis for V. Define a function $\phi\colon \mathbb{R}^r \to V$ by*

$$\phi(\lambda) = \lambda_1 v_1 + \cdots + \lambda_r v_r.$$

(If we need to emphasis the dependence on $\mathcal{V}$, we will write $\phi_{\mathcal{V}}$ rather than just ϕ.) Then there is an inverse function $\psi\colon V \to \mathbb{R}^r$ with $\phi(\psi(v)) = v$ for all $v \in V$, and $\psi(\phi(\lambda)) = \lambda$ for all $\lambda \in \mathbb{R}^d$. Moreover, both ϕ and ψ respect addition and scalar multiplication:

$$\begin{aligned} \phi(\lambda + \mu) &= \phi(\lambda) + \phi(\mu) & \qquad \phi(t\lambda) &= t\phi(\lambda) \\ \psi(v + w) &= \psi(v) + \psi(w) & \qquad \psi(tv) &= t\psi(v). \end{aligned}$$

Proof. By assumption the list $\mathcal{V}$ is linearly independent and $\operatorname{span}(\mathcal{V}) = V$. Consider an arbitrary vector $u \in V$. As $u \in \operatorname{span}(\mathcal{V})$ we can write u as a linear combination $u = \sum_i \lambda_i v_i$ say, which means that $u = \phi(\lambda)$ for some λ. We claim that this λ is unique. Indeed, if we also have $u = \phi(\mu) = \sum_i \mu_i v_i$ then we can subtract to get $\sum_i (\lambda_i - \mu_i) v_i = 0$. This is a linear relation on the list $\mathcal{V}$, but $\mathcal{V}$ is assumed to be independent, so it must be the trivial relation. This means that all the coefficients

$\lambda_i - \mu_i$ are zero, so $\lambda = \mu$ as required. It is now meaningful to define $\psi(u)$ to be the unique vector λ with $\psi(\lambda) = u$. We leave it as an exercise to check that this has all the stated properties. $\square$

Corollary 20.5. *Let V be a d-dimensional subspace of $\mathbb{R}^n$.*

(a) *Any linearly independent list in V has at most d elements.*
(b) *Any list that spans V has at least d elements.*
(c) *Any basis of V has exactly d elements.*
(d) *Any linearly independent list of length d in V is a basis.*
(e) *Any list of length d that spans V is a basis.*

Proof. Choose a linearly independent list $\mathcal{V}$ of length d in V. By Propositions 20.3 and 20.4, we see that $\mathcal{V}$ is a basis and gives rise to mutually inverse functions $\mathbb{R}^d \xrightarrow{\phi} V \xrightarrow{\psi} \mathbb{R}^d$. We can use these functions to transfer results that we have already proved for $\mathbb{R}^d$ and get corresponding results for V. Details are as follows.

(a) This is just the definition of $\dim(V)$.
(b) Let $\mathcal{W} = (w_1, \dots, w_r)$ be a list that spans V. We claim that the list $(\psi(w_1), \dots, \psi(w_r))$ spans $\mathbb{R}^d$. Indeed, for any $x \in \mathbb{R}^d$ we have $\phi(x) \in V$, and $\mathcal{W}$ spans V so $\phi(x) = \sum_j \mu_j w_j$ say. We can apply ψ to this to get

$$x = \psi(\phi(x)) = \psi(\sum_j \mu_j w_j) = \sum_j \mu_j \psi(w_j),$$

which expresses x as a linear combination of the vectors $\psi(w_j)$, as required. We know from Remark 9.9 that any list that spans $\mathbb{R}^d$ must have length at least d, so $r \geq d$ as claimed.
(c) This holds by combining (a) and (b).
(d) This is a restatement of Proposition 20.3.

(e) Let $\mathcal{W} = (w_1, \dots, w_d)$ be a list of length d that spans V. As in (b) we see that the list $(\psi(w_1), \dots, \psi(w_d))$ spans $\mathbb{R}^d$. By Proposition 10.12, this list is in fact a basis for $\mathbb{R}^d$, so in particular it is linearly independent. We claim that the original list $\mathcal{W}$ is also linearly independent. To see this, consider a linear relation $\sum_j \lambda_j w_j = 0$. By applying ψ to both sides, we get $\sum_i \lambda_j \psi(w_j) = 0$. As the vectors $\psi(w_j)$ are independent we see that $\lambda_j = 0$ for all j. This means that the original relation is the trivial one, as required. As $\mathcal{W}$ is linearly independent and spans V, it is a basis for V.

□

Proposition 20.6. *Let V be a subspace of $\mathbb{R}^n$. Then there is a unique RREF matrix B such that the columns of B^T form a basis for V. (We call this basis the* canonical basis *for V.)*

Remark 20.7. Except for the trivial case where $V = \{0\}$, any subspace V of $\mathbb{R}^n$ will have infinitely many different bases, but only one of them is the canonical basis. For example, take

$$V = \{\begin{bmatrix} x & y & z \end{bmatrix}^T \mid x + y + z = 0\}$$

and

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad a_2 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \quad b_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad b_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$$

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

These vectors all lie in V, because in each case the sum of the three components is zero. In fact one can check that the list

a_1, a_2 is a basis for V, but it is not the canonical basis, because the corresponding matrix

$$A = \left[\begin{array}{c} a_1^T \\ \hline a_2^T \end{array}\right] = \begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$

is not in RREF. Similarly, b_1, b_2 is another non-canonical basis for V. However, the list c_1, c_2 is the canonical basis for V.

Remark 20.8. As the canonical basis is uniquely defined, it gives an easy way to test whether two subspaces are the same, and it makes it simpler to define algorithms without any kind of ambiguity. However, one should not overemphasise the importance of the canonical basis. For many applications it may be more convenient to use a different basis.

We will first give the easy proof that B exists, then (after some preparation) the more complicated proof that it is unique.

Proof of existence. Let $\mathcal{U} = (u_1, \ldots, u_d)$ be any basis for V, and let A be the matrix with rows $u_1^T, \ldots, u_d^T$. Let B be the row-reduction of A, let $v_1^T, \ldots, v_d^T$ be the rows of B, and put $\mathcal{V} = (v_1, \ldots, v_d)$. Corollary 9.16 says that a row vector can be expressed as a linear combination of the rows of A if and only if it can be expressed as a linear combination of the rows of B. This implies that $\operatorname{span}(\mathcal{V}) = \operatorname{span}(\mathcal{U}) = V$. As $\mathcal{V}$ is a list of length d that spans the d-dimensional space V, we see that $\mathcal{V}$ is actually a basis for V. $\square$

Definition 20.9. Let $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ be a nonzero vector in $\mathbb{R}^n$. We say that x *starts in slot* k if $x_1, \ldots, x_{k-1}$ are zero, but x_k is not. Given a subspace $V \subseteq \mathbb{R}^n$, we say that k is a *jump* for V if there is a nonzero vector $x \in V$ that starts in slot k. We write $J(V)$ for the set of all jumps for V.

Example 20.10.

- The vector $\begin{bmatrix} 0 & 0 & 1 & 11 & 111 \end{bmatrix}^T$ starts in slot 3;
- The vector $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$ starts in slot 1;
- The vector $\begin{bmatrix} 0 & 0 & 0 & 0 & 0.1234 \end{bmatrix}^T$ starts in slot 5.

Example 20.11. Consider the subspace

$$V = \{\begin{bmatrix} s & -s & t+s & t-s \end{bmatrix}^T \mid s, t \in \mathbb{R}\} \subseteq \mathbb{R}^4.$$

If $s \neq 0$ then the vector $x = \begin{bmatrix} s & -s & t+s & t-s \end{bmatrix}^T$ starts in slot 1. If $s = 0$ but $t \neq 0$ then $x = \begin{bmatrix} 0 & 0 & t & t \end{bmatrix}^T$ and this starts in slot 3. If $s = t = 0$ then $x = 0$ and x does not start anywhere. Thus, the possible starting slots for x are 1 and 3, which means that $J(V) = \{1, 3\}$.

Example 20.12. Consider the subspace

$$W = \{\begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T \in \mathbb{R}^6 \mid a = b+c = d+e+f = 0\}.$$

Any vector $w = \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T$ in W can be written as $w = \begin{bmatrix} 0 & b & -b & d & e & -d-e \end{bmatrix}^T$, where b, d and e are arbitrary. If $b \neq 0$ then w starts in slot 2. If $b = 0$ but $d \neq 0$ then $w = \begin{bmatrix} 0 & 0 & 0 & d & e & -d-e \end{bmatrix}^T$ starts in slot 4. If $b = d = 0$ but $e \neq 0$ then $w = \begin{bmatrix} 0 & 0 & 0 & 0 & e & -e \end{bmatrix}^T$ starts in slot 5. If $b = d = e = 0$ then $w = 0$ and w does not start anywhere. Thus, the possible starting slots for w are 2, 4 and 5, so $J(W) = \{2, 4, 5\}$.

Lemma 20.13. *Let B be an RREF matrix, and suppose that the columns of B^T form a basis for a subspace $V \subseteq \mathbb{R}^n$. Then $J(V)$ is the same as the set of columns of B that contain pivots.*

Example proof. Rather than giving a formal proof, we will discuss an example that shows how the proof works. Consider the RREF matrix

$$B = \left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \end{array}\right] = \begin{bmatrix} 0 & 1 & \alpha & 0 & \beta & 0 & \gamma \\ 0 & 0 & 0 & 1 & \delta & 0 & \epsilon \\ 0 & 0 & 0 & 0 & 0 & 1 & \zeta \end{bmatrix}.$$

Put $V = \mathrm{span}(v_1, v_2, v_3) \subseteq \mathbb{R}^7$, so the vectors v_i (which are the columns of B^T) form a basis for V. There are pivots in columns 2, 4 and 6, so we need to show that $J(V) = \{2, 4, 6\}$. Any element $x \in V$ has the form

$$\begin{aligned} x &= \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \\ &= \begin{bmatrix} 0 & \lambda_1 & \lambda_1\alpha_1 & \lambda_2 & \lambda_1\beta + \lambda_2\delta & \lambda_3 & \lambda_1\gamma + \lambda 2\epsilon + \lambda_3\zeta \end{bmatrix}^T. \end{aligned}$$

Note that λ_k occurs on its own in the k'th pivot column, and all entries to the left of that involve only $\lambda_1, \dots, \lambda_{k-1}$. Thus, if $\lambda_1, \dots, \lambda_{k-1}$ are all zero but $\lambda_k \neq 0$ then x starts in the k'th pivot column. In more detail:

- If $\lambda_1 \neq 0$ then x has the form $x = \begin{bmatrix} 0 & \lambda_1 & * & * & * & * & * \end{bmatrix}^T$ and so x starts in slot 2 (the first pivot column).
- If $\lambda_1 = 0$ but $\lambda_2 \neq 0$ then x has the form $x = \begin{bmatrix} 0 & 0 & 0 & \lambda_2 & * & * & * \end{bmatrix}^T$ and so x starts in slot 4 (the second pivot column).
- If $\lambda_1 = \lambda_2 = 0$ but $\lambda_3 \neq 0$ then x has the form $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_3 & * \end{bmatrix}^T$ and so x starts in slot 6 (the third pivot column).
- If $\lambda_1 = \lambda_2 = \lambda_3 = 0$ then $x = 0$ and so x does not start anywhere.

Thus, the possible starting slots for nonzero vectors in V are just the same as the pivot columns. $\square$

We now return to the proof of Proposition 20.6.

Proof of uniqueness. Suppose we have a subspace $V \subseteq \mathbb{R}^n$ and two RREF matrices B and C such that the columns of B^T form a basis for V, and the columns of C^T also form a basis for V. We must show that $B = C$. First put $d = \dim(V)$. Using Corollary 20.5(c) we see that both B^T and C^T must have d columns, so both B and C are $d \times n$ matrices. Let $v_1, \dots, v_d$ be the columns of B and let $w_1, \dots, w_d$ be the columns of C. As the vectors v_i form a basis they must be nonzero, so each of the d rows of B is nonzero and therefore contains a single pivot, which means that B has precisely d pivot columns, say in columns $p_1, \dots, p_d$. We list these in order so that $p_1 < \dots < p_d$. Because B is in RREF, we see that the p_i'th component of v_j is one when $i = j$, and zero when $i \neq j$.

Similarly, the matrix C has precisely d pivot columns, say in columns $q_1, \dots, q_d$ with $q_1 < \dots < q_d$. Because C is in RREF, we see that the q_i'th component of w_j is one when $i = j$, and zero when $i \neq j$.

Next, using Lemma 20.13 we see that the pivot columns for B are the same as the jumps for V which are the same as the pivot columns for C. This means that $p_i = q_i$ for all i.

Now consider one of the vectors v_i. As $v_i \in V$ and $V = \operatorname{span}(w_1, \dots, w_d)$ we can write v_i as a linear combination of the vectors w_j, say $v_i = \lambda_1 w_1 + \dots + \lambda_d w_d$. Consider what this gives in the p_i'th column (or equivalently, the q_i'th column). On the left hand side we have 1, and on the right hand side the $\lambda_i w_i$ term contributes λ_i, and the other terms contribute nothing. We conclude that $\lambda_i = 1$. Now consider instead what we get in the p_j'th column, where $j \neq i$. On the left hand side we get zero, and on the right hand side the $\lambda_i w_i$ term contributes λ_i, and the other terms contribute nothing. We conclude that $\lambda_j = 0$ for $j \neq i$. The equation $v_i = \lambda_1 w_1 + \dots + \lambda_d w_d$ now simplifies to $v_i = w_i$. This holds for all i, so we have $B = C$ as claimed. $\square$

We next discuss how to find the canonical basis for a subspace $V \subseteq \mathbb{R}^n$. The method depends on how the subspace V is described in the first place. If V is given as the span of some list of vectors, then we proceed as follows.

Method 20.14. To find the canonical basis for a subspace $V = \operatorname{span}(v_1, \dots, v_r)$, we form the matrix

$$A = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_r^T \end{array}\right]$$

We then row-reduce to get an RREF matrix B, and discard any rows of zeros to get another RREF matrix C. The columns of C^T are then the canonical basis for V.

Proof of correctness. Note that C^T is obtained from B^T by discarding some columns of zeros, which does not affect the span. Thus, the span of the columns of C^T is the same as the span of the columns of B^T, and Corollary 9.16 tells us that this is the same as the span of the columns of A^T, which is V. Moreover, as each pivot column of C contains a single one, it is easy to see that the rows of C are linearly independent or equivalently the columns of C^T are linearly independent. As they are linearly independent and span V, they form a basis for V. As C is in RREF, this must be the canonical basis. $\square$

Example 20.15. Consider the plane

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x + y + z = 0 \right\}$$

as in Example 19.21. We showed there that $P = \operatorname{span}(u_1, u_2)$, where $u_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$. As the matrix

$$A = \left[\begin{array}{c} u_1^T \\ \hline u_2^T \end{array}\right] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

is already in RREF, we see that the list $\mathcal{U} = (u_1, u_2)$ is the canonical basis for P.

Example 20.16. Consider the subspace

$$V = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w+2x+3y+4z = 4w+3x+2y+z = 0\}$$

as in Example 19.22. We showed there that the vectors

$$c = \begin{bmatrix} -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \end{bmatrix}^T \qquad \text{and} \qquad d = \begin{bmatrix} -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \end{bmatrix}^T.$$

give a (non-canonical) basis for V. To find the canonical basis, we perform the following row-reduction:

$$\left[\begin{array}{c} c^T \\ \hline d^T \end{array}\right] = \begin{bmatrix} -\frac{2}{3} & 1 & 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & -2 & 1 \end{bmatrix}^T$ form the canonical basis for V.

We next need a method for finding the canonical basis for subspace when that subspace is originally given as an annihilator.

Method 20.17. Suppose that

$$V = \operatorname{ann}(u_1, \dots, u_r) = \{x \in \mathbb{R}^n \mid x.u_1 = \dots = x.u_r = 0\}.$$

To find the canonical basis for V:

(a) Write out the equations $x.u_r = 0, \dots, x.u_1 = 0$, listing the variables in backwards order (x_r down to x_1).
(b) Solve by row-reduction in the usual way (remembering to list the variables in backwards order throughout).
(c) Write the general solution as a sum of terms, each of which is an independent variable times a constant vector.
(d) These constant vectors form the canonical basis for V.

Remark 20.18. If we did not write the variables in reverse order, we would still get a basis for V, but it would not be the canonical basis. The corresponding matrix would not be row-reduced in the usual sense, but in a kind of mirror-image sense: the last nonzero entry in each row would be equal to one (rather than the first nonzero entry), and these ones would move to the left (rather than to the right).

Example 20.19. Put $V = \text{ann}(u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 9 \\ 13 \\ 5 \\ 3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 7 \\ 11 \\ 3 \\ 1 \end{bmatrix}.$$

The equations $x.u_3 = x.u_2 = x.u_1 = 0$ can be written as follows:

$$\begin{aligned} x_4 + 3x_3 + 11x_2 + 7x_1 &= 0 \\ x_4 + x_3 + x_2 + x_1 &= 0 \\ 3x_4 + 5x_3 + 13x_2 + 9x_1 &= 0. \end{aligned}$$

Note that we have written the variables in decreasing order, as specified in step (a) of the method. We can row-reduce the matrix of coefficients as follows:

$$\begin{bmatrix} 1 & 3 & 11 & 7 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 13 & 9 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 10 & 6 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 10 & 6 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & 5 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columns of this last matrix contain coefficients of x_4, x_3, x_2 and x_1 respectively. We conclude that our original system of equations is equivalent to the equations $x_4 - 4x_2 - 2x_1 = 0$ and $x_3 + 5x_2 + 3x_1 = 0$, which give $x_4 = 4x_2 + 2x_1$ and $x_3 = -5x_2 - 3x_1$. The independent variables are x_1 and x_2, and we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -5x_2 - 3x_1 \\ 4x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix}.$$

This can be written as $x_1 v_1 + x_2 v_2$, where $v_1 = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 0 & 1 & -5 & 4 \end{bmatrix}^T$. We conclude that (v_1, v_2) is the canonical basis for V.

Example 20.20. Let V be the set of all vectors $x \in \mathbb{R}^5$ that satisfy the equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 0 \\ x_1 + 2x_2 + 3x_3 + 3x_4 + 3x_5 &= 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 0. \end{aligned}$$

Equivalently, we can put

$$\begin{aligned} u_1 &= \begin{bmatrix}1 & 2 & 3 & 4 & 5\end{bmatrix}^T \\ u_2 &= \begin{bmatrix}1 & 2 & 3 & 3 & 3\end{bmatrix}^T \\ u_3 &= \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T. \end{aligned}$$

The three equations defining V are then equivalent to $u_1.x = 0$ and $u_2.x = 0$ and $u_3.x = 0$, so we can describe V as $\operatorname{ann}(u_1, u_2, u_3)$. To find the canonical basis, we rewrite the defining equations as follows:

$$\begin{aligned} x_5 + x_4 + x_3 + x_2 + x_1 &= 0 \\ 3x_5 + 3x_4 + 3x_3 + 2x_2 + x_1 &= 0 \\ 5x_5 + 4x_4 + 3x_3 + 2x_2 + x_1 &= 0. \end{aligned}$$

Here we have reversed the individual equations from left to right, and we have also moved the top equation to the bottom and the bottom equation to the top. This vertical reversal does not make any real difference, but later on it will make it easier to explain what we are doing in terms of matrices.

We now row-reduce the matrix of coefficients:

$$\begin{bmatrix}1&1&1&1&1\\3&3&3&2&1\\5&4&3&2&1\end{bmatrix} \to \begin{bmatrix}1&1&1&1&1\\0&0&0&-1&-2\\0&-1&-2&-3&-4\end{bmatrix} \to$$

$$\begin{bmatrix}1&1&1&1&1\\0&1&2&3&4\\0&0&0&1&2\end{bmatrix} \to \begin{bmatrix}1&1&1&0&-1\\0&1&2&0&-2\\0&0&0&1&2\end{bmatrix} \to$$

$$\begin{bmatrix}1&0&-1&0&1\\0&1&2&0&-2\\0&0&0&1&2\end{bmatrix}$$

The columns of this last matrix contain coefficients of x_5, x_4, x_3, x_2 and x_1 respectively. We conclude that our original system of equations is equivalent to the following system:

$$\begin{aligned} x_5 - x_3 + x_1 &= 0 \\ x_4 + 2x_3 - 2x_1 &= 0 \\ x_2 + 2x_1 &= 0. \end{aligned}$$

This gives $x_5 = x_3 - x_1$ and $x_4 = -2x_3 + 2x_1$ and $x_2 = -2x_1$, with x_1 and x_3 independent, so

$$\begin{aligned} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \\ x_3 \\ -2x_3 + 2x_1 \\ x_3 - x_1 \end{bmatrix} &= x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \\ &= x_1 v_1 + x_3 v_2 \text{ say.} \end{aligned}$$

It follows that the vectors

$$v_1 = \begin{bmatrix} 1 & -2 & 0 & 2 & 1 \end{bmatrix}^T \qquad \text{and} \qquad v_2 = \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \end{bmatrix}^T$$

form the canonical basis for V.

It is a slight defect of Method 20.17 that we translate backwards and forwards between equations and matrices more times than necessary. However, it is not too hard to formulate an equivalent method that works wholly with matrices.

The first ingredient is a process that is essentially the same as the standard method for writing down the solution to a system of equations that is already in RREF.

Method 20.21. Let B be an $m \times n$ matrix in RREF with no rows of zeros, and let $V \subseteq \mathbb{R}^n$ be the annihilator of the columns of B^T. We can find a basis for V as follows:

(a) The matrix B will have m pivots (one in each row). Let columns $p_1, \dots, p_m$ be the ones with pivots, and let columns $q_1, \dots, q_{n-m}$ be the ones without pivots.
(b) Delete the pivot columns from B to leave an $m \times (n-m)$ matrix, which we call C. Let the i'th row of C be c_i^T (so $c_i \in \mathbb{R}^{n-m}$ for $1 \leq i \leq m$).
(c) Now construct a new matrix D of shape $(n-m) \times n$ as follows: the p_i'th column is $-c_i$, and the q_j'th column is the standard basis vector e_j.
(d) The columns of D^T then form a basis for V.

Rather than proving formally that this method is valid, we will just show how it works out in an example that has all the features of the general case.

Example 20.22. Consider the case

$$B = \begin{bmatrix} 0 & 1 & a_3 & 0 & a_5 & a_6 & 0 & a_8 & 0 & a_{10} \\ 0 & 0 & 0 & 1 & b_5 & b_6 & 0 & b_8 & 0 & b_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_8 & 0 & c_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & d_{10} \end{bmatrix}$$

Here $m = 4$ and $n = 10$. The pivot columns (shown in blue) are $p_1 = 2$, $p_2 = 4$, $p_3 = 7$ and $p_4 = 9$. The non-pivot columns (shown in red) are $q_1 = 1$, $q_2 = 3$, $q_3 = 5$, $q_4 = 6$, $q_5 = 8$ and $q_6 = 10$. Deleting the pivot columns leaves the following matrix:

$$C = \left[\begin{array}{c} c_1^T \\ \hline c_2^T \\ \hline c_3^T \\ \hline c_4^T \end{array}\right] = \begin{bmatrix} 0 & a_3 & a_5 & a_6 & a_8 & a_{10} \\ 0 & 0 & b_5 & b_6 & b_8 & b_{10} \\ 0 & 0 & 0 & 0 & c_8 & c_{10} \\ 0 & 0 & 0 & 0 & 0 & d_{10} \end{bmatrix}.$$

We now form the matrix D. This is divided into red and blue columns in the same pattern as the original matrix B. In the red columns we have a spread-out copy of the identity matrix. In the blue columns we have the negatives of the transposes of the rows of C.

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_5 & 0 & -b_5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_6 & 0 & -b_6 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -a_8 & 0 & -b_8 & 0 & 0 & -c_8 & 1 & 0 & 0 \\ 0 & -a_{10} & 0 & -b_{10} & 0 & 0 & -c_{10} & 0 & -d_{10} & 1 \end{bmatrix}$$

The claim is thus that the following vectors (which are the columns of D^T) form a basis for V:

$$d_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad d_2 = \begin{bmatrix} 0 \\ -a_3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad d_3 = \begin{bmatrix} 0 \\ -a_5 \\ 0 \\ -b_5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$d_4 = \begin{bmatrix} 0 \\ -a_6 \\ 0 \\ -b_6 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad d_5 = \begin{bmatrix} 0 \\ -a_8 \\ 0 \\ -b_8 \\ 0 \\ 0 \\ -c_8 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad d_6 = \begin{bmatrix} 0 \\ -a_{10} \\ 0 \\ -b_{10} \\ 0 \\ 0 \\ -c_{10} \\ 0 \\ -d_{10} \\ 1 \end{bmatrix}$$

To see why this is true, we study V more directly. By definition, V is the set of all vectors $x = \begin{bmatrix} x_1 & \cdots & x_{10} \end{bmatrix} \in \mathbb{R}^{10}$ whose inner product with each of the columns of B^T is zero, which means that

$$\begin{array}{cccccccccc}
x_2 & +a_3x_3 & & +a_5x_5 & +a_6x_6 & & +a_8x_8 & & +a_{10}x_{10} & =0 \\
& & x_4 & +b_5x_5 & +b_6x_6 & & +b_8x_8 & & +b_{10}x_{10} & =0 \\
& & & & & x_7 & +c_8x_8 & & +c_{10}x_{10} & =0 \\
& & & & & & & x_9 & +d_{10}x_{10} & =0.
\end{array}$$

The variables x_2, x_4, x_7 and x_9 (corresponding to the pivot columns) are dependent, and the remaining variables x_1, x_3, x_5, x_6, x_8 and x_{10} (corresponding to the non-pivot columns) are independent. We can express the dependent variables in terms of the independent ones:

$$\begin{array}{ccccccccc}
x_2 = & -a_3x_3 & & -a_5x_5 & -a_6x_6 & & -a_8x_8 & & -a_{10}x_{10} \\
& & x_4 = & -b_5x_5 & -b_6x_6 & & -b_8x_8 & & -b_{10}x_{10} \\
& & & & & x_7 = & -c_8x_8 & & -c_{10}x_{10} \\
& & & & & & & x_9 = & -d_{10}x_{10}
\end{array}$$

This gives

$$x = \begin{bmatrix} x_1 \\ -a_3x_3 - a_5x_5 - a_6x_6 - a_8x_8 - a_{10}x_{10} \\ x_3 \\ -b_5x_5 - b_6x_6 - b_8x_8 - b_{10}x_{10} \\ x_5 \\ x_6 \\ -c_8x_8 - c_{10}x_{10} \\ x_8 \\ -d_{10}x_{10} \\ x_{10} \end{bmatrix} =$$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -a_3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -a_5 \\ 0 \\ -b_5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -a_6 \\ 0 \\ -b_6 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_8 \begin{bmatrix} 0 \\ -a_8 \\ 0 \\ -b_8 \\ 0 \\ 0 \\ -c_8 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_{10} \begin{bmatrix} 0 \\ -a_{10} \\ 0 \\ -b_{10} \\ 0 \\ 0 \\ -c_{10} \\ 0 \\ -d_{10} \\ 1 \end{bmatrix}$$

or in other words

$$x = x_1 d_1 + x_3 d_2 + x_5 d_3 + x_6 d_4 + x_8 d_5 + x_{10} d_{10}.$$

This proves that an arbitrary element $x \in V$ can be written (in a unique way) as a linear combination of the vectors d_i, so these vectors give a basis for V as required.

We can now describe a pure matrix algorithm to find the canonical basis of an annihilator.

Method 20.23. Let A be a $k \times n$ matrix, and let $V \subseteq \mathbb{R}^n$ be the annihilator of the columns of A^T. We can find the canonical basis for V as follows:

(a) Rotate A through $180°$ to get a matrix A^*.
(b) Row-reduce A^* and discard any rows of zeros to obtain a matrix B^* in RREF. This will have shape $m \times n$ for some m with $m \leq \min(k, n)$.
(c) The matrix B^* will have m pivots (one in each row). Let columns $p_1, \ldots, p_m$ be the ones with pivots, and let columns $q_1, \ldots, q_{n-m}$ be the ones without pivots.
(d) Delete the pivot columns from B^* to leave an $m \times (n - m)$ matrix, which we call C^*. Let the i'th row of C^* be c_i^T (so $c_i \in \mathbb{R}^{n-m}$ for $1 \leq i \leq m$).

(e) Now construct a new matrix D^* of shape $(n-m)\times n$ as follows: the p_i'th column is $-c_i$, and the q_j'th column is the standard basis vector e_j.
(f) Rotate D^* through $180°$ to get a matrix D.
(g) The columns of D^T then form the canonical basis for V.

Rather than giving a detailed proof that this is equivalent to Method 20.17, we will just explain some examples that should make the pattern clear.

Example 20.24. Put

$$u_1 = \begin{bmatrix} 9 \\ 13 \\ 5 \\ 3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 7 \\ 11 \\ 3 \\ 1 \end{bmatrix}$$

and $V = \operatorname{ann}(u_1, u_2, u_3)$ as in Example 20.19. The relevant matrix A for Method 20.17 is shown below, together with the matrix A^* obtained by rotating A.

$$A = \begin{bmatrix} 9 & 13 & 5 & 3 \\ 1 & 1 & 1 & 1 \\ 7 & 11 & 3 & 1 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1 & 3 & 11 & 7 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 13 & 9 \end{bmatrix}.$$

The matrix A^* is the same as the matrix of coefficients appearing in Example 20.19, and as we saw there we can row-reduce and delete zeros as follows:

$$A^* = \begin{bmatrix} 1 & 3 & 11 & 7 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 13 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & -2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} \color{blue}{1} & \color{blue}{0} & \color{red}{-4} & \color{red}{-2} \\ \color{blue}{0} & \color{blue}{1} & \color{red}{5} & \color{red}{3} \end{bmatrix} = B^*.$$

The pivot columns are $p_1 = 1$ and $p_2 = 2$, whereas the non-pivot columns are $q_1 = 3$ and $q_2 = 4$. We now delete the pivot columns to get

$$C^* = \left[\begin{array}{c} c_1^T \\ \hline c_2^T \end{array}\right] = \begin{bmatrix} -4 & -2 \\ 5 & 3 \end{bmatrix}.$$

Next, we construct the matrix D^*:

$$D^* = \left[\begin{array}{c|c|c|c} -c_1 & -c_2 & e_1 & e_2 \end{array}\right] = \begin{bmatrix} 4 & -5 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix}.$$

Finally, we rotate this through 180° to get

$$D = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \end{bmatrix}.$$

The canonical basis for V consists of the columns of D^T, namely $v_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ -5 \\ 4 \end{bmatrix}$. This is the same answer as we got in Example 20.19.

Example 20.25. Consider again the vectors

$$\begin{aligned} u_1 &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T \\ u_2 &= \begin{bmatrix} 1 & 2 & 3 & 3 & 3 \end{bmatrix}^T \\ u_3 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T. \end{aligned}$$

and the subspace $V = \operatorname{ann}(u_1, u_2, u_3)$ as in Example 20.20. The relevant matrix A for Method 20.23 is shown below, together with the matrix A^* obtained by rotating A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

Note that the rows of A^* are u_3 backwards, followed by u_2 backwards, followed by u_1 backwards. We could have used this rule to avoid having to write out A. Note also that A^* is the same as the matrix of coefficients appearing in Example 20.20. As we saw there, it row-reduces as follows:

$$A^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = B^*.$$

The matrix B^* has pivots in columns $p_1 = 1$ and $p_2 = 2$ and $p_3 = 4$; the remaining columns are $q_1 = 3$ and $q_2 = 5$. After deleting the pivot columns we are left with

$$C^* = \left[\begin{array}{c} c_1^T \\ \hline c_2^T \\ \hline c_3^T \end{array}\right] = \begin{bmatrix} -1 & -1 \\ 2 & -2 \\ 0 & 2 \end{bmatrix}$$

Next, we construct the matrix D^*:

$$D^* = \left[\begin{array}{c|c|c|c|c} -c_1 & -c_2 & e_1 & -c_3 & e_2 \end{array}\right] = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & 2 & 0 & -2 & 1 \end{bmatrix}.$$

Finally, we rotate this through $180°$ to get

$$D = \begin{bmatrix} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}.$$

The canonical basis for V consists of the columns of D^T, namely
$v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$. This is the same as the answer we obtained in Example 20.20.

If a subspace V is originally described as the annihilator of a list of vectors, then Method 20.17 gives us an alternative description of V as the span of a different list of vectors. We could also ask about the opposite problem: if V is originally described as a span, can we give an alternative description of V as an annihilator? It turns out that essentially the same method does the job (which is another manifestation of duality). The counterpart of Method 20.17 is as follows:

Method 20.26. Suppose that $V = \operatorname{span}(v_1, \dots, v_r)$.

(a) Write out the equations $x.v_r = 0, \dots, x.v_1 = 0$, listing the variables in backwards order (x_r down to x_1).
(b) Solve by row-reduction in the usual way.
(c) Write the general solution as a sum of terms, each of which is an independent variable times a constant vector.
(d) Call these constant vectors $u_1, \dots, u_s$. Then $V = \operatorname{ann}(u_1, \dots, u_s)$.

There is also a purely matrix-based version:

Method 20.27. Let A be a $k \times n$ matrix, and let $V \subseteq \mathbb{R}^n$ be the span of the columns of A^T. Construct a matrix D by the same steps as in Method 20.23. Then V is also the annihilator of the columns of D^T.

The proof of correctness is not especially hard, but we will omit it to save time.

21. Sums and intersections of subspaces

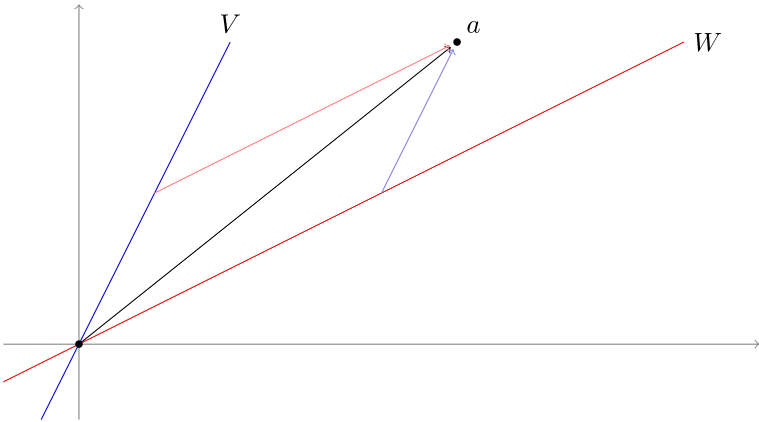
Definition 21.1. Let V and W be subspaces of $\mathbb{R}^n$. We define

$V + W = \{x \in \mathbb{R}^n \mid x$ can be expressed as $v + w$ for some $v \in V$

$V \cap W = \{x \in \mathbb{R}^n \mid x \in V$ and also $x \in W\}$.

Example 21.2. Put

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y = 2x \right\} \qquad W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid 2y = x \right\}$$



Then $V \cap W$ is the set of points lying on both lines, but the lines only meet at the origin, so $V \cap W = \{0\}$. On the other hand, it is clear that every point $a \in \mathbb{R}^2$ can be expressed as the sum of a point on V with a point on W, so $V + W = \mathbb{R}^2$. For an algebraic argument, consider an arbitrary point $a = \begin{bmatrix} x \\ y \end{bmatrix}$.

If we put

$$v = \frac{2y-x}{3}\begin{bmatrix}1\\2\end{bmatrix} \qquad\qquad w = \frac{2x-y}{3}\begin{bmatrix}2\\1\end{bmatrix}.$$

we find that $v \in V$ and $w \in W$ and $a = v + w$, which shows that $a \in V + W$ as claimed.

Example 21.3. Put

$$V = \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w = y \text{ and } x = z\}$$
$$W = \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w + z = x + y = 0\}.$$

For a vector $u = \begin{bmatrix}w & x & y & z\end{bmatrix}^T$ to lie in $V \cap W$ we must have $w = y$ and $x = z$ and $w = -z$ and $x = -y$. From this it follows easily that $u = \begin{bmatrix}w & -w & w & -w\end{bmatrix}^T$, so $V \cap W$ is just the set of multiples of $\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}$.

Now put

$$U = \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \mid w - x - y + z = 0\}$$
$$= \operatorname{ann}(\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^T).$$

We claim that $V + W = U$. To see this, consider a vector $u = \begin{bmatrix}w & x & y & z\end{bmatrix}^T$. In one direction, suppose that $u \in V + W$. This means that u can be written as $u = v + w$ for some $v \in V$ and $w \in W$. By inspecting the definition of V we see that $v = \begin{bmatrix}p & q & p & q\end{bmatrix}$ for some $p, q \in \mathbb{R}$. Similarly, by inspecting the definition of W we see that $w = \begin{bmatrix}-r & -s & s & r\end{bmatrix}^T$ for some $r, s \in \mathbb{R}$. This gives $u = \begin{bmatrix}p - r & q - s & p + s & q + r\end{bmatrix}$, so

$$w - x - y + z = (p - r) - (q - s) - (p + s) + (q + r) =$$
$$p - r - q + s - p - s + q + r = 0,$$

proving that $u \in U$ as required.

In the opposite direction, suppose we have $u \in U$, so $z = x + y - w$. Put $v = \begin{bmatrix}y & x & y & x\end{bmatrix}^T$ and $w = \begin{bmatrix}w - y & 0 & 0 & y - w\end{bmatrix}^T$.

We find that $v \in V$ and $w \in W$ and $v + w = u$, which proves that $u \in V + W$ as required.

If we take the methods of the previous section as given, then it is straightforward to calculate sums and intersections. The key point is as follows.

Proposition 21.4. *For any two lists $v_1, \ldots, v_r$ and $w_1, \ldots, w_s$ of vectors in $\mathbb{R}^n$, we have*

(a) $\operatorname{span}(v_1, \ldots, v_r) + \operatorname{span}(w_1, \ldots, w_s) = \operatorname{span}(v_1, \ldots, v_r, w_1, \ldots$
(b) $\operatorname{ann}(v_1, \ldots, v_r) \cap \operatorname{ann}(w_1, \ldots, w_s) = \operatorname{ann}(v_1, \ldots, v_r, w_1, \ldots$

Proof. (a) An arbitrary element $x \in \operatorname{span}(v_1, \ldots, v_r) + \operatorname{span}(w_1, \ldots, w_s)$ has the form $x = v + w$, where v is an arbitrary element of $\operatorname{span}(v_1, \ldots, v_r)$ and w is an arbitrary element of $\operatorname{span}(w_1, \ldots, w_s)$. This means that $v = \sum_{i=1}^{r} \lambda_i v_i$ and $w = \sum_{j=1}^{s} \mu_j w_j$ for some coefficients $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_s$, so

$$x = \lambda_1 v_1 + \cdots + \lambda_r v_r + \mu_1 w_1 + \cdots + \mu_s w_s.$$

This is precisely the same as the general form for an element of $\operatorname{span}(v_1, \ldots, v_r, w_1, \ldots, w_s)$.

(b) A vector $x \in \mathbb{R}^n$ lies in $\operatorname{ann}(v_1, \ldots, v_r)$ if and only if $x.v_1 = \cdots = x.v_r = 0$. Similarly, x lies in $\operatorname{ann}(w_1, \ldots, w_s)$ iff $x.w_1 = \cdots = x.w_s$. Thus, x lies in $\operatorname{ann}(v_1, \ldots, v_r) \cap \operatorname{ann}(w_1, \cdots, w_s)$ iff both sets of equations are satisfied, or in other words

$$x.v_1 = \cdots = x.v_r = x.w_1 = \cdots = x.w_s = 0.$$

This is precisely the condition for x to lie in $\operatorname{ann}(v_1, \ldots, v_r, w_1, \ldots, w_s)$.

$\square$

This justifies the following methods:

Method 21.5. To find the sum of two subspaces $V, W \subseteq \mathbb{R}^n$:

(a) Find a list $\mathcal{V}$ such that $V = \text{span}(\mathcal{V})$. It may be that V is given to us as the span of some list (possibly slightly disguised, as in Examples 19.16 and 19.17, or Proposition 19.19), in which case there is nothing to do. Alternatively, if V is given to as as the annihilator of some list, then we can use Method 20.17 to find a basis $\mathcal{V}$ for V, which in particular will have $\text{span}(\mathcal{V}) = V$.
(b) Find a list $\mathcal{W}$ such that $W = \text{span}(\mathcal{W})$ (in the same way).
(c) Now $V + W$ is the span of the combined list $\mathcal{V}, \mathcal{W}$. We can thus use Method 20.14 to find the canonical basis for $V + W$ if desired.

Method 21.6. To find the intersection of two subspaces $V, W \subseteq \mathbb{R}^n$:

(a) Find a list $\mathcal{V}'$ such that $V = \text{ann}(\mathcal{V}')$. It may be that V is given to us as the annihilator of some list (possibly slightly disguised as in Examples 19.10 and 19.11, or Proposition 19.14), in which case there is nothing to do. Alternatively, if V is given to as as the span of some list, then we can use Method 20.26 to find a list $\mathcal{V}'$ such that $\text{ann}(\mathcal{V}') = V$.
(b) Find a list $\mathcal{W}'$ such that $W = \text{ann}(\mathcal{W}')$ (in the same way).
(c) Now $V \cap W$ is the annihilator of the combined list $\mathcal{V}', \mathcal{W}'$. This list can again be made canonical by row-reduction if required.

Remark 21.7. The dimensions of V, W, $V \cap W$ and $V + W$ are linked by the following important formula:

$$\dim(V \cap W) + \dim(V + W) = \dim(V) + \dim(W).$$

Thus, if we know three of these four dimensions, we can rearrange the formula to find the fourth one. Alternatively, if you believe that you have found bases for V, W, $V \cap W$ and $V + W$, you can use the formula as a check that your bases are correct.

It is not too hard to prove the formula, but it requires some ideas that are a little different from those that we are emphasising in this course, so we will omit it.

Example 21.8. Put

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad a = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

and $V = \operatorname{span}(v_1, v_2, v_3)$ and $W = \operatorname{span}(w_1, w_2, w_3)$. We first claim that $V + W = \mathbb{R}^4$. The systematic proof is by Method 21.5 (which uses $V + W = \operatorname{span}(v_1, v_2, v_3, w_1, w_2, w_3)$):

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline v_3^T \\ \hline w_1^T \\ \hline w_2^T \\ \hline w_3^T \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c} e_1^T \\ \hline e_2^T \\ \hline e_3^T \\ \hline e_4^T \\ \hline 0 \\ \hline 0 \end{array}\right]$$

The conclusion is that (e_1, e_2, e_3, e_4) is the canonical basis for $V + W$, or in other words $V + W = \mathbb{R}^4$ as claimed. For a less systematic but more efficient argument, we can note that

$$e_1 = v_1 \qquad e_2 = w_1 - v_1 \qquad e_3 = v_2 - w_1 \qquad e_4 = v_3 - v_2.$$

It follows that e_1, e_2, e_3 and e_4 are all elements of $V + W$, and thus that $V + W = \mathbb{R}^4$.

We now want to determine $V \cap W$. The first step is to describe V and W as annihilators rather than spans, which we can do using Method 20.26. For the case of V, we write down the equations $x.v_3 = 0$, $x.v_2 = 0$ and $x.v_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 + x_2 + x_1 &= 0 \\ x_3 + x_2 + x_1 &= 0 \\ x_1 &= 0. \end{aligned}$$

Clearly we have $x_1 = x_4 = 0$ and $x_3 = -x_2$, with x_2 arbitrary. In other words, we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ -x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = x_2 a.$$

We conclude that $V = \operatorname{ann}(a)$.

For the case of W, we write down the equations $x.w_3 = 0$, $x.w_2 = 0$ and $x.w_1 = 0$, with the variables x_i in descending order:

$$\begin{aligned} x_4 + x_3 &= 0 \\ x_3 + x_2 &= 0 \\ x_2 + x_1 &= 0. \end{aligned}$$

This easily gives $x_4 = -x_3 = x_2 = -x_1$, so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = x_1 b.$$

We conclude that $W = \operatorname{ann}(b)$.

We now have $V \cap W = \operatorname{ann}(a, b)$. To find the canonical basis for this, we write the equations $x.b = 0$ and $x.a = 0$, again with the variables in decreasing order:

$$\begin{aligned} -x_4 + x_3 - x_2 + x_1 &= 0 \\ -x_3 + x_2 &= 0 \end{aligned}$$

After row-reduction we get $x_4 = x_1$ and $x_3 = x_2$ with x_1 and x_2 arbitrary. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We conclude that the vectors $u_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for $V \cap W$. As a sanity check we can note that

$$u_1 = v_1 - v_2 + v_3 \in V \qquad\qquad u_2 = v_2 - v_1 \in V$$
$$u_1 = w_1 - w_2 + w_3 \in W \qquad\qquad u_2 = w_2 \in W.$$

These equations show directly that u_1 and u_2 lie in $V \cap W$.

Remark 21.9. We can use Remark 21.7 to check our work in Example 21.2. The list (v_1, v_2, v_3) is easily seen to be linearly independent, and by definition it spans V, so we have $\dim(V) = 3$. Similarly $\dim(W) = 3$. We showed that $V + W = \mathbb{R}^4$, so $\dim(V + W) = 4$. We also a produced vectors u_1 and u_2 that form a basis for $V \cap W$, so $\dim(V \cap W) = 2$. As expected, we have

$$\dim(V+W)+\dim(V\cap W) = 4+2 = 6 = 3+3 = \dim(V)+\dim(W).$$

Example 21.10. Consider the vectors

$$v_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \qquad w_1 = \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \qquad w_2 = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

and the subspaces $V = \operatorname{span}(v_1, v_2)$ and $W = \operatorname{span}(w_1, w_2)$. We will find the canonical bases for V, W, $V + W$ and $V \cap W$. The first three are straightforward row-reductions. For V we have

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4\end{bmatrix} \to$$

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\end{bmatrix}$$

We conclude that the vectors $v_1' = \begin{bmatrix}1 & 0 & -1 & -2\end{bmatrix}^T$ and $v_2' = \begin{bmatrix}0 & 1 & 2 & 3\end{bmatrix}^T$ form the canonical basis for V. Similarly, the row-reduction

$$\begin{bmatrix} w_1^T \\ \hline w_2^T \end{bmatrix} = \begin{bmatrix} -3 & -1 & 1 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} -3 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

shows that the vectors $w_1' = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$ and $w_2' = [0 \quad 1 \quad -1$ form the canonical basis for W.

We next want to find the canonical basis for $V + W$. As $V = \operatorname{span}(v_1, v_2) = \operatorname{span}(v_1', v_2')$ and $W = \operatorname{span}(w_1, w_2) = \operatorname{span}(w_1', w_2')$ we have

$$V + W = \operatorname{span}(v_1, v_2, w_1, w_2) = \operatorname{span}(v_1', v_2', w_1', w_2').$$

We could find the canonical basis by row-reducing either the matrix $[v_1|v_2|w_1|w_2]^T$ or the matrix $[v_1'|v_2'|w_1'|w_2']^T$. The latter will involve less work, as it is closer to RREF in the first place:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that the following vectors form the canonical basis for $V + W$:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

In particular, we have $\dim(V + W) = 3$.

Next, to understand $V \cap W$, we need to write V and W as annihilators rather than as spans. For W this is easy: we just put $b_1 = \begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T$ and $b_2 = \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$, and after considering the form of the vectors w_1' and w_2' we see that

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ -x_2 \\ -x_1 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_1 + x_4 = x_2 + x_3 = 0 \right\}$$

For V, we note that the equations $x.v_1' = 0$ and $x.v_2' = 0$ are

$$-2x_4 - x_3 + x_1 = 0$$
$$3x_4 + 2x_3 + x_2 = 0.$$

We can solve these in the usual way to get

$$x_3 = -2x_2 - 3x_1$$
$$x_4 = x_2 + 2x_1$$

(with x_2 and x_1 arbitrary), so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_2 - 3x_1 \\ x_2 + 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

From this we can deduce that $V = \operatorname{ann}(a_1, a_2)$, where $a_1 = \begin{bmatrix}1 & 0 & -3 & 2\end{bmatrix}^T$ and $a_2 = \begin{bmatrix}0 & 1 & -2 & 1\end{bmatrix}^T$. We now have

$$V \cap W = \operatorname{ann}(a_1, a_2) \cap \operatorname{ann}(b_1, b_2) = \operatorname{ann}(a_1, a_2, b_1, b_2).$$

To find the canonical basis for this, we solve the equations $x.b_2 = x.b_1 = x.a_2 = x.a_1 = 0$:

$$x_3 + x_2 = 0$$
$$x_4 + x_1 = 0$$
$$x_4 - 2x_3 + x_2 = 0$$
$$2x_4 - 3x_3 + x_1 = 0.$$

The first two equations give $x_3 = -x_2$ and $x_4 = -x_1$, which we can substitute into the remaining equations to get $x_2 = x_1/3$. This leads to $x = x_1 \begin{bmatrix} 1 & 1/3 & -1/3 & -1 \end{bmatrix}^T$, so the vector $c = \begin{bmatrix} 1 & 1/3 & -1/3 & -1 \end{bmatrix}^T$ is (by itself) the canonical basis for $V \cap W$. In particular, we have $\dim(V \cap W) = 1$

As a check, we note that

$$\dim(V+W)+\dim(V\cap W) = 3+1 = 2+2 = \dim(V)+\dim(W),$$

as expected.

22. Rank and normal form

Let A be an $m \times n$ matrix.

- In Definition 19.18 we defined $\operatorname{img}(A)$ to be the set of vectors $v \in \mathbb{R}^m$ that can be written as $v = Au$ for some $u \in \mathbb{R}^n$.
- In Definition 19.13 we defined $\ker(A)$ to be the set of vectors $u \in \mathbb{R}^n$ such that $Au = 0$.
- In Proposition 19.19 we showed that $\operatorname{img}(A)$ is the span of the columns of A. In particular, this means that $\operatorname{img}(A)$ is a subspace of $\mathbb{R}^m$.
- In Proposition 19.14 we showed that $\ker(A)$ is the annihilator of the transposed rows of A. In particular, this means that $\ker(A)$ is a subspace of $\mathbb{R}^n$.

Definition 22.1. For any matrix A, the *rank* of A is the dimension of $\operatorname{img}(A)$, and the *nullity* of A is the dimension of $\ker(A)$. We write $\operatorname{rank}(A)$ for the rank and (A) for the nullity.

As the columns of A are essentially the same as the rows of A^T, we see that $\operatorname{rank}(A)$ is also the dimension of the span of the rows of A^T. In this section we will repeatedly need to go back and forth between rows and columns, so we need to introduce some new terminology.

Definition 22.2. A matrix A is in *reduced column echelon form* (RCEF) if A^T is in RREF, or equivalently:

RCEF0: Any column of zeros come at the right hand end of the matrix, after all the nonzero columns.
RCEF1: In any nonzero column, the first nonzero entry is equal to one. These entries are called *copivots*.
RCEF2: In any nonzero column, the copivot is further down than the copivots in all previous columns.
RCEF3: If a column contains a copivot, then all other entries in that column are zero.

Definition 22.3. Let A be a matrix. The following operations on A are called *elementary column operations*:

ECO1: Exchange two columns.
ECO2: Multiply a column by a nonzero constant.
ECO3: Add a multiple of one column to another column.

Proposition 22.4. *If a matrix A is in RCEF, then the rank of A is just the number of nonzero columns.*

Proof. Let the nonzero columns be $u_1, \dots, u_r$, and put $U = \operatorname{span}(u_1, \dots, u_r)$. This is the same as the span of *all* the columns, because columns of zeros do not contribute anything to the span. We claim that the vectors u_i are linearly independent. To see this, note that each u_i contains a copivot, say in the q_i'th row. As the matrix is in RCEF we have $q_1 < \cdots < q_r$, and the q_i'th row is all zero apart from the copivot in u_i. In other words, for $j \neq i$ the q_i'th entry in u_j is zero. Now suppose we have a linear relation $\lambda_1 u_1 + \cdots + \lambda_r u_r = 0$. By looking at the q_i'th entry, we see that λ_i is zero. This holds for all i, so we have the trivial linear relation. This proves that the list $u_1, \dots, u_r$ is linearly independent, so it forms a basis for U, so $\dim(U) = r$. We thus have $\operatorname{rank}(A) = r$ as claimed. $\square$

The following two results can be proved in the same way as their counterparts for row operations; we will not spell out the details.

Proposition 22.5. *Any matrix A can be converted to RCEF by a sequence of elementary column operations.*

Proof. Analogous to Method 6.3. □

Proposition 22.6. *Suppose that A can be converted to B by a sequence of elementary column operations. Then $B = AV$ for some invertible matrix V.*

Proof. It is clear that A^T can be converted to B^T by a series of elementary row operations corresponding to the column operations that were used to convert A to B. Thus, Corollary 11.10 tells us that $B^T = UA^T$ for some invertible matrix U. We thus have $B = B^{TT} = (UA^T)^T = A^{TT}U^T = AU^T$. Here U^T is invertible by Remark 11.7, so we can take $V = U^T$. □

Proposition 22.7. *Suppose that A can be converted to B by a sequence of elementary column operations. Then the span of the columns of A is the same as the span of the columns of B (and so $\operatorname{rank}(A) = \operatorname{rank}(B)$).*

Proof. Analogous to Corollary 9.16. □

The next result is a little more subtle:

Proposition 22.8. *Suppose that A can be converted to B by a sequence of elementary* row *operations. Then* $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Proof. Let the columns of A be $v_1, \dots, v_n$ and put $V = \operatorname{span}(v_1, \dots$ so $\operatorname{rank}(A) = \dim(V)$. Corollary 11.10 tells us that there is an invertible matrix P such that

$$B = PA = P\left[\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array}\right] = \left[\begin{array}{c|c|c} Pv_1 & \cdots & Pv_n \end{array}\right],$$

so the vectors Pv_i are the columns of B. Thus, if we put $W = \operatorname{span}(Pv_1, \dots, Pv_n)$, then $\operatorname{rank}(B) = \dim(W)$.

We next claim that if $x \in V$ then $Px \in W$. Indeed, if $x \in V$ then x must be a linear combination of the vectors v_i, say $x = \sum_{i=1}^n \lambda_i v_i$ for some sequence of coefficients $\lambda_1, \dots, \lambda_n$. This means that $Px = \sum_{i=1}^n \lambda_i P v_i$, which is a linear combination of $Pv_1, \dots, Pv_n$, so $Px \in W$.

Similarly, we claim that if $y \in W$ then $P^{-1}y \in V$. Indeed, if $y \in W$ then y must be a linear combination of the vectors Pv_i, say $y = \sum_{i=1}^n \lambda_i P v_i$ for some sequence of coefficients $\lambda_1, \dots, \lambda_n$. This means that $P^{-1}y = \sum_{i=1}^n \lambda_i P^{-1} P v_i = \sum_{i=1}^n \lambda_i v_i$, which is a linear combination of $v_1, \dots, v_n$, so $P^{-1}y \in V$.

Now choose a basis $a_1, \dots, a_r$ for V (so $\operatorname{rank}(A) = \dim(V) = r$). We claim that the vectors $Pa_1, \dots, Pa_r$ form a basis for W. Indeed, we just showed that $Px \in W$ whenever $x \in V$, so the vectors Pa_i are at least elements of W. Consider an arbitrary element $y \in W$. We then have $P^{-1}y \in V$, but the vectors a_i form a basis for V, so we have $P^{-1}y = \sum_{i=1}^r \mu_i a_i$ for some sequence of coefficients μ_i. This means that $y = PP^{-1}y = \sum_i \mu_i P a_i$, which expresses y as a linear combination of the vectors Pa_i. It follows that the list $Pa_1, \dots, Pa_r$ spans W. We need to check that it is also linearly independent. Suppose we have a linear relation $\sum_i \lambda_i P a_i = 0$. After multiplying by P^{-1}, we get a linear relation $\sum_i \lambda_i a_i = 0$. The list $a_1, \dots, a_r$ is assumed to be a basis for V, so this must be the trivial relation, so $\lambda_1 = \cdots = \lambda_r = 0$, or in other words the original relation $\sum_i \lambda_i P a_i = 0$ was the trivial one. We have now shown that

$Pa_1, \dots, Pa_r$ is a basis for W, so $\dim(W) = r$. In conclusion, we have $\operatorname{rank}(A) = r = \operatorname{rank}(B)$ as required. $\square$

Definition 22.9. An $n \times m$ matrix A is in *normal form* if it has the form

$$A = \left[\begin{array}{c|c} I_r & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right]$$

for some r. (The case $r = 0$ is allowed, in which case A is just the zero matrix.)

Note that if A is in normal form as above, then the rank of A is r, which is the number of ones in A.

Example 22.10. There are precisely four different 3×5 matrices that are in normal form, one of each rank from 0 to 3 inclusive.

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Proposition 22.11. *Any $n \times m$ matrix A can be converted to a matrix C in normal form by a sequence of row and column operations. Moreover:*

(a) *There is an invertible $n \times n$ matrix U and an invertible $m \times m$ matrix V such that $C = UAV$.*
(b) *The rank of A is the same as the rank of C, which is the number of ones in C.*

Proof. We first perform row operations to get a matrix B in RREF. By Corollary 11.10 there is an invertible matrix U such that $B = UA$. (This has to be an $n \times n$ matrix for the product UA to make sense.) Now subtract multiples of the pivot columns from the columns further to the right. As each pivot column contains nothing other than the pivot, the only effect of these column operations is to set everything to the right of a pivot equal to zero. However, every nonzero entry in B is either a pivot or to the right of a pivot, so after these operation we just have the pivots from B and everything else is zero. We now just move all columns of zeros to the right hand end, which leaves a matrix C in normal form. As C was obtained from B by a sequence of elementary column operations, we have $C = BV$ for some invertible $m \times m$ matrix V. As $B = UA$, it follows that $C = UAV$. Propositions 22.7 and 22.8 tell us that neither row nor column operations affect the rank, so $\text{rank}(A) = \text{rank}(C)$, and because C is in normal form, $\text{rank}(C)$ is just the number of ones in C. □

Example 22.12. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix}.$$

This can be row-reduced as follows:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 6 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now perform column operations:

$$\begin{bmatrix}1 & 3 & 0 & 1\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \to$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

(Subtract column 1 from column 4, and 3 times column 1 from column 2; subtract 4 times column 3 from column 4; exchange columns 2 and 3.) We are left with a matrix of rank 2 in normal form, so $\operatorname{rank}(A) = 2$.

Example 22.13. Consider the matrix

$$A = \begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 3 & 4 & 5\\ 4 & 5 & 6\end{bmatrix}$$

This can be reduced to normal form as follows:

$$A \to \begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ 0 & -2 & -4\\ 0 & -3 & -6\end{bmatrix} \to \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & -2 & -4\\ 0 & -3 & -6\end{bmatrix} \to$$

$$\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Subtract multiples of row 1 from the other rows; multiply row 2 by -1; subtract multiples of row 2 from the other rows; add column 1 to column 3; subtract 2 times column 2 from column 3.) The final matrix has rank 2, so we must also have $\text{rank}(A) = 2$.

Proposition 22.14. *For any matrix A we have* $\text{rank}(A) = \text{rank}(A^T)$

Proof. As in Proposition 22.11, we can convert A by row and column operations to a matrix C in normal form, and $\text{rank}(A)$ is the number of ones in C. If we transpose everything then the row operations become column operations and *vice-versa*, so A^T can be converted to C^T by row and column operations, and C^T is also in normal form, so $\text{rank}(A^T)$ is the number of ones in C^T. This is clearly the same as the number of ones in C, so $\text{rank}(A) = \text{rank}(A^T)$. □

Remark 22.15. Some authors define the *row rank* of A to be the dimension of the span of the rows, and the *column rank* to be the dimension of the span of the columns. In our terminology, the column rank is just $\text{rank}(A)$ and the row rank is $\text{rank}(A^T)$. Thus, the above proposition says that the row rank is the same as the column rank.

Corollary 22.16. *If A is an $n \times m$ matrix. Then* $\text{rank}(A) \leq \min(n, m)$.

Proof. Let V be the span of the columns of A, and let W be the span of the columns of A^T. Now V is a subspace of $\mathbb{R}^n$, so $\dim(V) \leq n$, but W is a subspace of $\mathbb{R}^m$, so $\dim(W) \leq m$. On the other hand, Proposition 22.14 tells us that $\dim(V) =$

$\dim(W) = \mathrm{rank}(A)$, so we have $\mathrm{rank}(A) \leq n$ and also $\mathrm{rank}(A) \leq m$, so $\mathrm{rank}(A) \leq \min(n, m)$. □

23. Orthogonal and symmetric matrices

Definition 23.1. Let A be an $n \times n$ matrix. We say that A is an *orthogonal matrix* it is invertible and $A^{-1} = A^T$.

Definition 23.2. Let $v_1, \dots, v_r$ be a list of r vectors in $\mathbb{R}^n$. We say that this list is *orthonormal* if $v_i.v_i = 1$ for all i, and $v_i.v_j = 0$ whenever i and j are different.

Remark 23.3. Suppose we have vectors v_1, v_2 and v_3 in $\mathbb{R}^3$, where everything has a familiar geometric interpretation. The equation $v_i.v_i = 1$ then means that v_1, v_2 and v_3 are unit vectors, and the equation $v_i.v_j = 0$ means that v_1, v_2 and v_3 are orthogonal to each other.

Proposition 23.4. *Any orthonormal list of length n in $\mathbb{R}^n$ is a basis.*

Proof. Let $v_1, \dots, v_n$ be an orthonormal list of length n. Suppose we have a linear relation $\sum_{i=1}^n \lambda_i v_i = 0$. We can take the dot product of both sides with v_p to get $\sum_{i=1}^n \lambda_i (v_i.v_p) = 0$. Now most of the terms $v_i.v_p$ are zero, because of the assumption that $v_i.v_j = 0$ whenever $i \neq j$. After dropping the terms where $i \neq p$, we are left with $\lambda_p(v_p.v_p) = 0$. Here $v_p.v_p = 1$ (by the definition of orthonormality) so we just have $\lambda_p = 0$. This works for all p, so our linear relation is the trivial one. This proves that the list $v_1, \dots, v_n$ is linearly independent. A linearly independent list of n vectors in $\mathbb{R}^n$ is automatically a basis by Proposition 10.12. □

Definitions 23.1 and 23.2 are really two different ways of looking at the same thing:

Proposition 23.5. *Let A be an $n \times n$ matrix. Then A is an orthogonal matrix if and only if the columns of A form an orthonormal list.*

Proof. By definition, A is orthogonal if and only if A^T is an inverse for A, or in other words $A^T A = I_n$. Let the columns of A be $v_1, \dots, v_n$. Then

$$A^T A = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_n^T \end{array}\right] \left[\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array}\right] = \begin{bmatrix} v_1.v_1 & \cdots & v_1.v_n \\ \vdots & \ddots & \vdots \\ v_n.v_1 & \cdots & v_n.v_n \end{bmatrix}$$

In other words, the entry in the (i, j) position in $A^T A$ is just the dot product $v_i.v_j$. For $A^T A$ to be the identity we need the diagonal entries $v_i.v_i$ to be one, and the off-diagonal entries $v_i.v_j$ (with $i \neq j$) to be zero. This means precisely that the list $v_1, \dots, v_n$ is orthonormal. $\square$

Definition 23.6. Let A be an $n \times n$ matrix, with entries a_{ij}. We say that A is *symmetric* if $A^T = A$, or equivalently $a_{ij} = a_{ji}$ for all i and j.

Example 23.7. A 4×4 matrix is symmetric if and only if it has the form

$$\begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}.$$

Example 23.8. The matrices A and B are symmetric, but C and D are not.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 111 & 11 & 1 \\ 11 & 111 & 11 \\ 1 & 11 & 111 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 10 & 1000 \\ 1 & 10 & 1000 \\ 1 & 10 & 1000 \end{bmatrix}$$

Lemma 23.9. *Let A be an $n \times n$ matrix, and let u and v be vectors in $\mathbb{R}^n$. Then $u.(Av) = (A^T u).v$. In particular, if A is symmetric then $u.(Av) = (Au).v$.*

Proof. Put $p = A^T u$ and $q = Av$, so the claim is that $u.q = p.v$. By the definition of matrix multiplication, we have $q_i = \sum_j A_{ij} v_j$, so $u.q = \sum_i u_i q_i = \sum_{i,j} u_i A_{ij} v_j$. Similarly, we have $p_j = \sum_i (A^T)_{ji} u_i$, but $(A^T)_{ji} = A_{ij}$ so $p_j = \sum_i u_i A_{ij}$. It follows that $p.v = \sum_j p_j v_j = \sum_{i,j} u_i A_{ij} v_j$, which is the same as $u.q$, as claimed.

Alternatively, we can recall that for $x, y \in \mathbb{R}^n$ the dot product $x.y$ can be interpreted as the matrix product $x^T y$. Thus $(Au).v = (Au)^T v$, but $(Au)^T = u^T A^T$ (by Proposition 3.4) so $(Au).v = u^T (A^T v) = u.(A^T v)$. □

Proposition 23.10. *Let A be an $n \times n$ symmetric matrix (with real entries).*

(a) *All eigenvalues of A are real numbers.*
(b) *If u and v are (real) eigenvectors for A with distinct eigenvalues, then u and v are orthogonal.*

Proof. (a) Let λ be a complex eigenvalue of A, say $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. We must show that $\beta = 0$, so that λ is actually a real number. As λ is an eigenvalue, there is a nonzero vector u with $Au = \lambda u$. We must allow for the possibility that u does not have real entries; we let v and w be the real and imaginary parts, so $v, w \in \mathbb{R}^n$ and $u = v + iw$. We now have

$$Av+iAw = A(v+iw) = Au = \lambda u = (\alpha+i\beta)(v+iw) = (\alpha v-\beta w)+i$$

As the entries in A are real, we see that the vectors Av and Aw are real. We can thus compare real and imaginary parts in the above equation to get

$$\begin{aligned} Av &= \alpha v - \beta w \\ Aw &= \beta v + \alpha w. \end{aligned}$$

From this we get

$$\begin{aligned} (Av).w &= \alpha v.w - \beta w.w \\ v.(Aw) &= \beta v.v + \alpha v.w. \end{aligned}$$

However, A is symmetric, so $(Av).w = v.(Aw)$ by Lemma 23.9. After a little rearrangement this gives $\beta(v.v + w.w) = 0$. Now $v.v$ is the sum of the squares of the entries in v, and similarly for w. By assumption $u \neq 0$, so at least one of v and w must be nonzero, so $v.v+w.w > 0$. We can thus divide by $v.v+w.w$ to get $\beta = 0$ and $\lambda = \alpha \in \mathbb{R}$ as claimed.

(b) Now suppose instead that u and v are eigenvectors of A with distinct eigenvalues, say λ and μ. By assumption we have $Au = \lambda u$ and $Av = \mu v$ and $\lambda \neq \mu$. As A is symmetric we have $(Au).v = u.(Av)$. As $Au = \lambda u$ and $Av = \mu v$ this becomes $\lambda\, u.v = \mu\, u.v$, so $(\lambda-\mu)u.v = 0$. As $\lambda \neq \mu$ we can divide by $\lambda - \mu$ to get $u.v = 0$, which means that u and v are orthogonal.

□

Remark 23.11. In the case $n = 2$ we can argue more directly. A 2×2 symmetric matrix has the form $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, so $\chi_A(t) = t^2 - (a+d)t + (ad - b^2)$. The eigenvalues are

$$\lambda = (a + d \pm \sqrt{(a+d)^2 - 4(ad - b^2)})/2.$$

The expression under the square root is

$$(a+d)^2 - 4(ad - b^2) = a^2 + 2ad + d^2 - 4ad + 4b^2 =$$

$$a^2 - 2ad + d^2 + 4b^2 = (a-d)^2 + (2b)^2.$$

This is the sum of two squares, so it is nonnegative, so the square root is real, so the two eigenvalues are both real.

Proposition 23.12. *Let A be an $n \times n$ symmetric matrix. Then there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for A.*

Partial proof. We will show that the Theorem holds whenever A has n distinct eigenvalues. In fact it is true even without that assumption, but the proof is harder.

Let the eigenvalues of A be $\lambda_1, \dots, \lambda_n$. For each i we choose an eigenvector u_i of eigenvalue λ_i. As u_i is an eigenvector we have $u_i \neq 0$ and so $u_i.u_i > 0$, so we can define $v_i = u_i/\sqrt{u_i.u_i}$. This is just a real number times u_i, so it is again an eigenvector of eigenvalue λ_i. It satisfies

$$v_i.v_i = \frac{u_i.u_i}{\sqrt{u_i.u_i}\,\sqrt{u_i.u_i}} = 1$$

(so it is a unit vector). For $i \neq j$ we have $v_i.v_j = 0$ by Proposition 23.10(b). This shows that the sequence $v_1, \dots, v_n$ is orthonormal, and it is automatically a basis by Proposition 23.4 (or alternatively, by Proposition 13.22). $\square$

Remark 23.13. Let A be an $n \times n$ symmetric matrix again. The characteristic polynomial $\chi_A(t)$ has degree n, so by well-known properties of polynomials it can be factored as $\chi_A(t) = \prod_{i=1}^{n}(\lambda_i - t)$ for some complex numbers $\lambda_1, \dots, \lambda_n$. By Proposition 23.10(a) these are in fact all real. Some of them might be the same, but that would be a concidence which could only happen if the matrix A was very simple or had some kind of hidden symmetry. Thus, our proof of Proposition 23.12 covers almost all cases (but some of the cases that are not covered are the most interesting ones).

Example 23.14. Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(which appeared on one of the problem sheets) and the vectors

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$u_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad u_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

These satisfy $Au_1 = Au_2 = Au_3 = Au_4 = 0$ and $Au_5 = 5u_5$, so they are eigenvectors of eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

and $\lambda_5 = 5$. Because λ_5 is different from $\lambda_1, \dots, \lambda_4$, Proposition 23.10(b) tells us that u_5 must be orthogonal to $u_1, \dots, u_4$, and indeed it is easy to see directly that $u_1.u_5 = \dots = u_4.u_5 = 0$. However, the eigenvectors $u_1, \dots, u_4$ all share the same eigenvalue so there is no reason for them to be orthogonal and in fact they are not: we have

$$u_1.u_2 = u_1.u_3 = u_1.u_4 = u_2.u_3 = u_2.u_4 = u_3.u_4 = 1.$$

However, it is possible to choose a different basis of eigenvectors where all the eigenvectors are orthogonal to each other. One such choice is as follows:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \\ 0 \end{bmatrix}$$

$$v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -4 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

It is easy to check directly that

$$Av_1 = Av_2 = Av_3 = Av_4 = 0 \qquad\qquad Av_5 = 5v_5$$

$$v_1.v_2 = v_1.v_3 = v_1.v_4 = v_1.v_5 = v_2.v_3 = v_2.v_4 = v_2.v_5 = v_3.v_4 = v$$

so the v_i are eigenvectors and are orthogonal to each other. However, they are not orthonormal, because

$$v_1.v_1 = 2 \qquad v_2.v_2 = 6 \qquad v_3.v_3 = 12 \qquad v_4.v_4 = 20$$

This is easily fixed: if we put

$$w_1 = \frac{v_1}{\sqrt{2}} \qquad w_2 = \frac{v_2}{\sqrt{6}} \qquad w_3 = \frac{v_3}{\sqrt{12}} \qquad w_4 = \frac{v_4}{\sqrt{20}}$$

then $w_1, \ldots, w_5$ is an orthonormal basis for $\mathbb{R}^5$ consisting of eigenvectors for A.

Corollary 23.15. *Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal matrix U and a diagonal matrix D such that $A = UDU^T = UDU^{-1}$.*

Proof. Choose an orthonormal basis of eigenvectors $u_1, \ldots, u_n$, and let λ_i be the eigenvalue of u_i. Put $U = [u_1|\cdots|u_n]$ and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Proposition 14.4 tells us that $U^{-1}AU = D$ and so $A = UDU^{-1}$. Proposition 23.5 tells us that U is an orthogonal matrix, so $U^{-1} = U^T$. □

Example 23.16. Let A be the 5×5 matrix in which every entry is one, as in Example 23.14. Following the prescription in the above proof, we put

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & -3/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & 0 & -4/\sqrt{20} & 1/\sqrt{5} \end{bmatrix} \qquad D$$

The general theory tells us that $A = UDU^T$. We can check this directly:

$$UD = \begin{bmatrix} * & * & * & * & 1/\sqrt{5} \\ * & * & * & * & 1/\sqrt{5} \\ * & * & * & * & 1/\sqrt{5} \\ * & * & * & * & 1/\sqrt{5} \\ * & * & * & * & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} \end{bmatrix}$$

$$UDU^T = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(To save space we have written stars for values that are irrelevant because they get multiplied by zero.)

Example 23.17. Write $\rho = \sqrt{3}$ for brevity (so $\rho^2 = 3$), and consider the symmetric matrix

$$A = \begin{bmatrix} 0 & 1 & \rho \\ 1 & 0 & -\rho \\ \rho & -\rho & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}
\chi_A(t) &= \det \begin{bmatrix} -t & 1 & \rho \\ 1 & -t & -\rho \\ \rho & -\rho & -t \end{bmatrix} \\
&= -t \det \begin{bmatrix} -t & -\rho \\ -\rho & -t \end{bmatrix} - \det \begin{bmatrix} 1 & -\rho \\ \rho & -t \end{bmatrix} + \rho \det \begin{bmatrix} 1 & -t \\ \rho & -\rho \end{bmatrix} \\
&= -t(t^2 - \rho^2) - (-t + \rho^2) + \rho(-\rho + t\rho) \\
&= -t^3 + 3t + t - 3 - 3 + 3t \\
&= -t^3 + 7t - 6 = -(t-1)(t-2)(t+3).
\end{aligned}$$

It follows that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -3$. Eigenvectors can be found by row-reduction:

$$\begin{aligned}
A - I &= \begin{bmatrix} -1 & 1 & \rho \\ 1 & -1 & -\rho \\ \rho & -\rho & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -\rho \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
A - 2I &= \begin{bmatrix} -2 & 1 & \rho \\ 1 & -2 & -\rho \\ \rho & -\rho & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -\rho \\ 0 & -3 & -\rho \\ 0 & \rho & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\rho/3 \\ 0 & 1 & \rho/3 \\ 0 & 0 & 0 \end{bmatrix} \\
A + 3I &= \begin{bmatrix} 3 & 1 & \rho \\ 1 & 3 & -\rho \\ \rho & -\rho & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -\rho \\ 0 & -8 & 4\rho \\ 0 & -4\rho & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \rho/2 \\ 0 & 1 & -\rho/2 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

From this we can read off the following eigenvectors:

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} \rho/3 \\ -\rho/3 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -\rho/2 \\ \rho/2 \\ 1 \end{bmatrix}.$$

Because the matrix A is symmetric and the eigenvalues are distinct, it is automatic that the eigenvectors u_i are orthogonal to each other. However, they are not normalised: instead we have

$$\begin{aligned} u_1.u_1 &= 1^2 + 1^2 = 2 \\ u_2.u_2 &= (\rho/3)^2 + (-\rho/3)^2 + 1^2 = 1/3 + 1/3 + 1 = 5/3 \\ u_3.u_3 &= (-\rho/2)^2 + (\rho/2)^2 + 1^2 = 3/4 + 3/4 + 1 = 5/2. \end{aligned}$$

The vectors $v_i = u_i/\sqrt{u_i.u_i}$ form an orthonormal basis of eigenvectors. Explicitly, this works out as follows:

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \\ \sqrt{3/5} \end{bmatrix} \qquad v_3 = \begin{bmatrix} -\sqrt{3/10} \\ \sqrt{3/10} \\ \sqrt{2/5} \end{bmatrix}.$$

It follows that if we put

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{5} & -\sqrt{3/10} \\ 1/\sqrt{2} & -1/\sqrt{5} & \sqrt{3/10} \\ 0 & \sqrt{3/5} & \sqrt{2/5} \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

then U is an orthogonal matrix and $A = UDU^T$.

Corollary 23.18. *Let A be an $n \times n$ symmetric matrix, and suppose that all the eigenvalues of A are positive. Then there is a symmetric matrix B such that $A = B^2$.*

Proof. Let U and D be as in the previous corollary. In particular, $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, where the λ_i are the eigenvalues and so $\lambda_i \geq 0$. We can thus put $E = \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ and $B = UEU^T$. It is clear that $E^T = E$, and it follows that

$$B^T = (UEU^T)^T = U^{TT}E^TU^T = UEU^T = B.$$

Moreover, as $U^T = U^{-1}$ we have $U^TU = I_n$ and

$$B^2 = UEU^TUEU^T = UEEU^T = UDU^T = A.$$

□

Definition 23.19. (a) A *linear form* on $\mathbb{R}^n$ is a function of the form $L(x) = \sum_{i=1}^n a_ix_i$ (for some constants $a_1, \ldots, a_n$.

(b) A *quadratic form* on $\mathbb{R}^n$ is a function of the form $Q(x) = \sum_{i=1}^n \sum_{j=1}^n b_{ij}x_ix_j$ for some constants b_{ij}.

Example 23.20. (a) We can define a linear form on $\mathbb{R}^3$ by $L(x) = 7x_1 - 8x_2 + 9x_3$.

(b) We can define a quadratic form on $\mathbb{R}^4$ by $Q(x) = 10x_1x_2 + 12x_3x_4 - 14x_1x_4 - 16x_2x_3$.

Remark 23.21. Given a linear form $L(x) = \sum_i a_ix_i$, we can form the vector $a = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^T$, and clearly $L(x) = a.x = a^Tx$.

Remark 23.22. Now suppose instead that we have a quadratic form $Q(x) = \sum_{i,j} b_{ij}x_ix_j$. We can then form the matrix B with entries b_{ij}, and we find that $Q(x) = x^TBx$. For example, if $n = 2$ and $Q(x) = x_1^2 + 4x_1x_2 + 7x_2^2$ then $B = \begin{bmatrix} 1 & 4 \\ 0 & 7 \end{bmatrix}$ and

$$\begin{aligned} x^TBx &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 + 4x_2 \\ 7x_2 \end{bmatrix} \\ &= x_1^2 + 4x_1x_2 + 7x_2^2 = Q(x). \end{aligned}$$

However, there is a slight ambiguity in this construction. As $x_1x_2 = x_2x_1$, we could equally well describe $Q(x)$ as $Q(x) =$

$x_1^2 + x_1x_2 + 3x_2x_1 + 7x_2^2$. This would give a different matrix B, namely $B = \begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix}$, but it would still be true that $Q(x) = x^TBx$. A third possibility would be to describe $Q(x)$ as $x_1^2 + 2x_1x_2 + 2x_2x_1 + 7x_2^2$, which gives $B = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$, and yet again $Q(x) = x^TBx$. In this last case we have "shared the coefficient equally" between x_1x_2 and x_2x_1, so the matrix B is symmetric. It is clear that we can do this for any quadratic form, and that this eliminates any ambiguity.

For example, we considered above the quadratic form

$$Q(x) = 10x_1x_2 + 12x_3x_4 - 14x_1x_4 - 16x_2x_3.$$

This can be rewritten symmetrically as

$$Q(x) = 5x_1x_2+5x_2x_1+6x_3x_4+6x_4x_3-7x_1x_4-7x_4x_1-8x_2x_3-8x_3$$

which corresponds to the symmetric matrix

$$B = \begin{bmatrix} 0 & 5 & 0 & -7 \\ 5 & 0 & -8 & 0 \\ 0 & -8 & 0 & 6 \\ -7 & 0 & 6 & 0 \end{bmatrix}$$

Proposition 23.23. *Let $Q(x)$ be a quadratic form on $\mathbb{R}^n$. Then there are integers $r, s \geq 0$ and nonzero vectors $v_1, \ldots, v_r, w_1, \ldots, w_s$ such that all the v's and w's are orthogonal to each other, and*

$$Q(x) = (x.v_1)^2 + \cdots + (x.v_r)^2 - (x.w_1)^2 - \cdots - (x.w_s)^2.$$

In other words, if we define linear forms L_i and M_j by $L_i(x) = x.v_i$ and $M_j(x) = x.w_j$ then

$$Q = L_1^2 + \cdots + L_r^2 - M_1^2 - \cdots - M_s^2.$$

The rank *of Q is defined to be $r + s$, and the* signature *is defined to be $r - s$.*

Proof. As explained in Remark 23.22, there is a unique symmetric matrix B such that $Q(x) = x^T Bx$. By Proposition 23.12, we can find an orthonormal basis $u_1, \dots, u_n$ for $\mathbb{R}^n$ such that each u_i is an eigenvector for B, with eigenvalue λ_i say. Let r be the number of indices i for which $\lambda_i > 0$, and let s be the number of indices i for which $\lambda_i < 0$. We can assume that the eigenvalues and eigenvectors have been ordered such that $\lambda_1, \dots, \lambda_r > 0$ and $\lambda_{r+1}, \dots, \lambda_{r+s} < 0$ and any eigenvalues after λ_{r+s} are zero. Now put $U = [u_1 | \cdots | u_n]$ and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We have seen that $B = UDU^T$, so

$$Q(x) = x^T Bx = x^T UDU^T x = (U^T x)^T (DU^T x) = (U^T x).(DU^T x)$$

Now

$$U^T x = \left[\begin{array}{c} u_1^T \\ \hline \vdots \\ \hline u_n^T \end{array}\right] x = \begin{bmatrix} u_1.x \\ \vdots \\ u_n.x \end{bmatrix}$$

$$DU^T x = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \begin{bmatrix} u_1.x \\ \vdots \\ u_n.x \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1.x \\ \vdots \\ \lambda_n u_n.x \end{bmatrix}$$

$$\begin{aligned} Q(x) = (U^T x).(DU^T x) &= \begin{bmatrix} u_1.x \\ \vdots \\ u_n.x \end{bmatrix} . \begin{bmatrix} \lambda_1 u_1.x \\ \vdots \\ \lambda_n u_n.x \end{bmatrix} \\ &= \lambda_1 (u_1.x)^2 + \cdots + \lambda_n (u_n.x)^2. \end{aligned}$$

We now group these terms according to whether λ_i is positive, negative or zero. For $1 \leq i \leq r$ we have $\lambda_i > 0$ and we put $v_i = \sqrt{\lambda_i} u_i$ so $\lambda_i (u_i.x)^2 = (v_i.x)^2$. For $r+1 \leq i \leq r+s$ we have $\lambda_i < 0$ and we put $w_{i-r} = \sqrt{|\lambda_i|} u_i$ so $\lambda_i (u_i.x)^2 = -(w_{i-r}.x)^2$.

For $i > r+s$ we have $\lambda_i = 0$ and $\lambda_i(u_i.x)^2 = 0$. We thus have

$$Q(x) = (x.v_1)^2 + \cdots + (x.v_r)^2 - (x.w_1)^2 - \cdots - (x.w_s)^2$$

as required. $\square$

Example 23.24. Consider the quadratic form $Q(x) = x_1x_2 - x_3x_4$ on $\mathbb{R}^4$. It is elementary that for all $a, b \in \mathbb{R}$ we have

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2.$$

Using this, we can rewrite $Q(x)$ as

$$Q(x) = \left(\frac{x_1+x_2}{2}\right)^2 - \left(\frac{x_1-x_2}{2}\right)^2 - \left(\frac{x_3+x_4}{2}\right)^2 + \left(\frac{x_3-x_4}{2}\right)^2.$$

Now put

$$v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \end{bmatrix} \quad w_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

It is straightforward to check that these are all orthogonal to each other, and we have $Q(x) = (x.v_1)^2 + (x.v_2)^2 - (x.w_1)^2 - (x.w_2)^2$.

Example 23.25. Consider the quadratic form $Q(x) = 4x_1x_2 + 6x_2x_3 + 4x_3x_4$ on $\mathbb{R}^4$. This can be rewritten symmetrically as

$$Q(x) = 2x_1x_2 + 2x_2x_1 + 3x_2x_3 + 3x_3x_2 + 2x_3x_4 + 2x_4x_3,$$

so the corresponding matrix is

$$B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

We can find the characteristic polynomial as follows:

$$\chi_B(t) = \det\begin{bmatrix} -t & 2 & 0 & 0 \\ 2 & -t & 3 & 0 \\ 0 & 3 & -t & 2 \\ 0 & 0 & 2 & -t \end{bmatrix}$$

$$= -t\det\begin{bmatrix} -t & 3 & 0 \\ 3 & -t & 2 \\ 0 & 2 & -t \end{bmatrix} - 2\det\begin{bmatrix} 2 & 3 & 0 \\ 0 & -t & 2 \\ 0 & 2 & -t \end{bmatrix}$$

$$\det\begin{bmatrix} -t & 3 & 0 \\ 3 & -t & 2 \\ 0 & 2 & -t \end{bmatrix} = -t(t^2-4) - 3(-3t) = 13t - t^3$$

$$\det\begin{bmatrix} 2 & 3 & 0 \\ 0 & -t & 2 \\ 0 & t & -2 \end{bmatrix} = 2(t^2-4) - 3(0-0) = 2t^2 - 8$$

$$\begin{aligned} \chi_B(t) &= -t(13t - t^3) - 2(2t^2 - 8) = t^4 - 17t^2 + 16 \\ &= (t^2-1)(t^2-16) \\ &= (t-1)(t+1)(t-4)(t+4). \end{aligned}$$

This shows that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$ and $\lambda_3 = -1$ and $\lambda_4 = -4$. By row-reducing the matrices $B - \lambda_i I$, we find the corresponding eigenvectors as follows:

$$t_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad t_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad t_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad t_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$$

In each case we see that $t_i.t_i = 10$ so the corresponding orthonormal basis consists of the vectors $u_i = t_i/\sqrt{10}$. Following the prescription in Proposition 23.23, we now put

$$\begin{aligned}
v_1 &= \sqrt{\lambda_1} u_1 &&= t_1/\sqrt{10} = \sqrt{1/10}\begin{bmatrix} 2 & 1 & -1 & -2 \end{bmatrix}^T \\
v_2 &= \sqrt{\lambda_2} u_2 &&= \sqrt{4} t_2/\sqrt{10} = \sqrt{2/5}\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T \\
w_1 &= \sqrt{|\lambda_3|} u_3 &&= t_3/\sqrt{10} = \sqrt{1/10}\begin{bmatrix} 2 & -1 & -1 & 2 \end{bmatrix}^T \\
w_2 &= \sqrt{|\lambda_4|} u_4 &&= \sqrt{4} t_4/\sqrt{10} = \sqrt{2/5}\begin{bmatrix} 1 & -2 & 2 & -1 \end{bmatrix}^T
\end{aligned}$$

We conclude that

$$Q(x) = (x.v_1)^2 + (x.v_2)^2 - (x.w_1)^2 - (x.w_2)^2.$$

Appendix A. List of all methods

This list contains all the methods of calculation that are explicitly laid out as 'Methods' in the main text. Some other methods are implicit in the examples.

- Method 5.4: Solve a system of linear equations that is already in RREF.
- Method 6.3: Reduce a matrix to RREF by row operations.
- Method 6.9: Solve a general system of linear equations.
- Method 13.14: Find the eigenvalues and eigenvectors of a square matrix.
- Method 7.6: Express a vector v as a linear combination of a given list of vectors $u_1, \dots, u_r$ (or show that this is impossible).
- Method 8.8: Check whether a given list of vectors is linearly dependent or linearly independent.
- Method 9.7: Check whether a given list of vectors in $\mathbb{R}^n$ spans all of $\mathbb{R}^n$.
- Method 10.5: Check whether a given list of vectors in $\mathbb{R}^n$ forms a basis of $\mathbb{R}^n$.

- Method 10.8: Express a vector in terms of a given basis for $\mathbb{R}^n$.
- Method 20.14: Find the canonical basis for $\mathrm{span}(v_1, \ldots, v_r$
- Method 20.17: Find the canonical basis for $\mathrm{ann}(u_1, \ldots, u_s)$
- Method 20.23: version of Method 20.17 formulated purely in terms of matrices.
- Method 20.26: Given vectors $v_1, \ldots, v_r$, find vectors $u_1, \ldots, u_s$ such that $\mathrm{span}(v_1, \ldots, v_r) = \mathrm{ann}(u_1, \ldots, u_s)$.
- Method 20.27: version of Method 20.26 formulated purely in terms of matrices.
- Method 21.5: Given subspaces $V, W \subseteq \mathbb{R}^n$, find the canonical basis for $V + W$.
- Method 21.6: Given subspaces $V, W \subseteq \mathbb{R}^n$, find the canonical basis for $V \cap W$.
- Method 11.11: Find the inverse of a square matrix (if it exists).
- Method 12.9: Find the determinant of a square matrix by row reduction.

Appendix B. Determinants

Note: the material in this appendix is not examinable.

Definition B.1. A *permutation* of the set $N = \{1, \ldots, n\}$ is a function $\sigma \colon N \to N$ that has an inverse. Equivalently, a function $\sigma \colon N \to N$ is a permutation if for every $j \in N$ there is a unique $i \in N$ such that $\sigma(i) = j$.

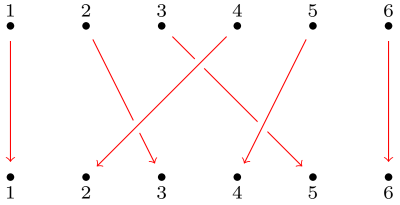
Example B.2. In the case $n = 6$, we have a permutation σ given by

$$\sigma(1) = 1 \qquad \sigma(2) = 3 \qquad \sigma(3) = 5 \qquad \sigma(4) = 2 \qquad \sigma(5) = 4$$

More compactly, we can describe σ by just listing the values:

$$\sigma = \langle 1, 3, 5, 2, 4, 6 \rangle.$$

This can be displayed as a picture:



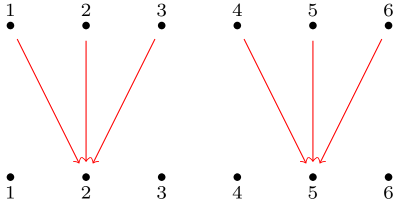
Example B.3. In the case $n = 6$, we can define a function $\theta\colon N \to N$ by

$$\theta(1) = 2 \qquad \theta(2) = 2 \qquad \theta(3) = 2 \qquad \theta(4) = 5 \qquad \theta(5) = 5 \qquad \theta$$

More compactly, we can describe θ by just listing the values:

$$\theta = \langle 2, 2, 2, 5, 5, 5 \rangle.$$

It can be displayed as a picture:



The function θ is **not** a permutation, because it has no inverse. For a permutation there would have to be a unique number i with $\theta(i) = 5$. In fact there are three different possibilities for i, namely $i = 4$, $i = 5$ or $i = 6$. There would also have to be a unique number j with $\theta(j) = 6$. In fact, there are no possibilities for j.

Definition B.4. Let $\sigma\colon N \to N$ be a permutation. A *reversal* for σ is a pair (i, j) of numbers in N such that $i < j$ but

$\sigma(i) > \sigma(j)$. We put

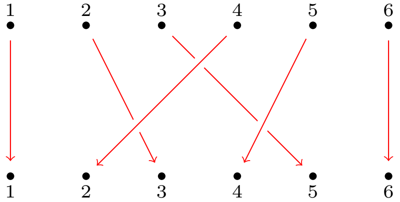
$$\begin{aligned} L(\sigma) &= \text{ the set of reversals for } \sigma \\ l(\sigma) = |L(\sigma)| &= \text{ the number of reversals for } \sigma \\ \operatorname{sgn}(\sigma) &= (-1)^{l(\sigma)}. \end{aligned}$$

We call $\operatorname{sgn}(\sigma)$ the *signature* of σ. We say that σ is an *even permutation* if the number $l(\sigma)$ is even, or equivalently $\operatorname{sgn}(\sigma) = +1$. We say that σ is an *odd permutation* if the number $l(\sigma)$ is odd, or equivalently $\operatorname{sgn}(\sigma) = -1$.

Example B.5. Consider the permutation given by

$$\sigma(1) = 1 \qquad \sigma(2) = 3 \qquad \sigma(3) = 5 \qquad \sigma(4) = 2 \qquad \sigma(5) = 4$$

as in Example B.2.

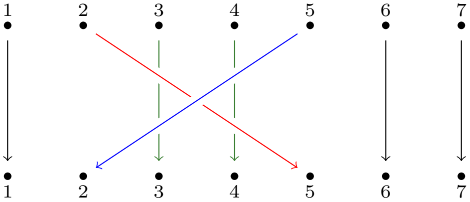


We have $2 < 4$ but $\sigma(2) = 5 > \sigma(4) = 2$, so the pair $(2, 4)$ is a reversal for σ. In terms of the picture, this corresponds to the fact that the line starting at 2 crosses over the line starting at 4. Similarly:

- The line starting at 3 crosses the line starting at 4, showing that the pair $(3, 4)$ is a reversal. More explicitly, we have $3 < 4$ and $\sigma(3) = 5 > \sigma(4) = 2$.
- The line starting at 3 crosses the line starting at 5, showing that the pair $(3, 5)$ is a reversal. More explicitly, we have $3 < 5$ and $\sigma(3) = 5 > \sigma(5) = 4$.

This is a complete list of all the reversals, so $L(\sigma) = \{(2,4),(3,4),(3,5)\}$, so $l(\sigma) = 3$ and $\mathrm{sgn}(\sigma) = (-1)^3 = -1$, showing that σ is an odd permutation.

Example B.6. Suppose we have $1 \leq p < q \leq n$, and we let $\tau\colon N \to N$ be the permutation given by $\tau(p) = q$ and $\tau(q) = p$ and $\tau(i) = i$ for all $i \neq p, q$. The case where $n = 7$ and $p = 2$ and $q = 5$ can be displayed as follows:



There are three types of reversals:

(a) For each i with $p < i < q$, the pair (p, i) is a reversal. (These correspond to the places where the sloping red line crosses the vertical green lines.)

(b) For each i with $p < i < q$, the pair (i, q) is a reversal. (These correspond to the places where the sloping blue line crosses the vertical green lines.)

(c) The pair (p, q) is a reversal, corresponding to the place where the red and blue lines cross.

Clearly the number of reversals of type (a) is the same as the number of reversals of type (b), so with the single reversal of type (c) we get an odd number of reversals altogether. This shows that τ is an odd permutation, or in other words $\mathrm{sgn}(\tau) = -1$.

Definition B.7. Let A be an $n \times n$ matrix, and let a_{ij} denote the entry in the i'th row of the j'th column. We define

$$\det(A) = \sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

We first check that this is the same as the traditional definition for 2×2 and 3×3 matrices.

Example B.8. Consider a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. In this case there are only two possible permutations: the identity permutation ι given by $\iota(1) = 1$ and $\iota(2) = 2$, and the transposition τ given by $\tau(1) = 2$ and $\tau(2) = 1$. The identity has no reversals, so $l(\iota) = 0$ and $\operatorname{sgn}(\iota) = 1$. The pair $(1, 2)$ is a reversal for τ, and it is the only one, so $l(\tau) = 1$ and $\sigma(\tau) = -1$. Definition B.7 therefore gives

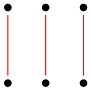
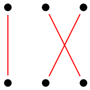
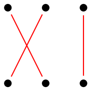
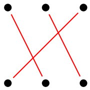
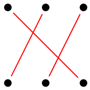
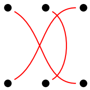
$$\begin{aligned} \det(A) &= \operatorname{sgn}(\iota) a_{1,\iota(1)} a_{2,\iota(2)} + \operatorname{sgn}(\tau) a_{1,\tau(1)} a_{2,\tau(2)} \\ &= a_{11}a_{22} - a_{12}a_{21}, \end{aligned}$$

which is the same as the traditional definition.

Example B.9. Consider a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{13} \end{bmatrix}$$

The determinant is a sum over all the possible permutations of the set $N = \{1, 2, 3\}$. These can be tabulated as follows:

	perm	picture	reversals	sig	term
σ_1	$\langle 1, 2, 3 \rangle$		0	$+1$	$+a_{11}a_{22}a_{33}$
σ_2	$\langle 1, 3, 2 \rangle$		1	-1	$-a_{11}a_{23}a_{32}$
σ_3	$\langle 2, 1, 3 \rangle$		1	-1	$-a_{12}a_{21}a_{33}$
σ_4	$\langle 2, 3, 1 \rangle$		2	$+1$	$+a_{12}a_{23}a_{31}$
σ_5	$\langle 3, 1, 2 \rangle$		2	$+1$	$+a_{13}a_{21}a_{32}$
σ_6	$\langle 3, 2, 1 \rangle$		3	-1	$-a_{13}a_{22}a_{31}$

By adding up the terms in the last column, we get

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$+a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

This is easily seen to be the same as the traditional formula:

$$
\begin{aligned}
&\det(A)\\
=&a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\\
=&a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a\\
=&a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{1}
\end{aligned}
$$

We can also calculate the determinant of a diagonal matrix the identity the definitions.

Proposition B.10. *Let A be a diagonal matrix, so the entries a_{ij} are zero for $i \neq j$. Then the determinant is just the product of the entries on the diagonal:*

$$\det(A) = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

In particular, for the identity matrix I_n we have $\det(I_n) = 1$.

Proof. The formula for $\det(A)$ has a term $\operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}$ for each permutation σ. If σ is not the identity permutation then for some i we will have $i \neq \sigma(i)$, so the factor $a_{i,\sigma(i)}$ will be zero, so the whole product will be zero. Thus, we only have a (potentially) nonzero term when σ is the identity, in which case $\operatorname{sgn}(\sigma) = 1$ and the term is just $\prod_{i=1}^{n} a_{ii}$. In the case of the identity matrix the diagonal elements a_{ii} are all equal to one, so we just get $\det(I_n) = 1$. $\square$

In fact, this can be strengthened as follows:

Proposition B.11. *Let A be a lower triangular matrix, so the entries a_{ij} are zero for $i < j$. Then the determinant is just the product of the entries on the diagonal:*

$$\det(A) = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

The same also holds if A is upper triangular.

Proof. We will prove the lower triangular case, and leave the upper triangular case to the reader.

The formula for $\det(A)$ has a term $\operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}$ for each permutation σ. This can only be nonzero if all the factors $a_{i,\sigma(i)}$ are on or below the diagonal, which means that $\sigma(i) \leq i$ for all i. In particular we must have $\sigma(1) \leq 1$, but $\sigma(1)$ is certainly in the set $N = \{1, \dots, n\}$, so we must have $\sigma(1) = 1$. Next, we must have $\sigma(2) \leq 2$ so $\sigma(2) \in \{1, 2\}$. However, we already have $\sigma(1) = 1$ and σ is a permutation so no value can be repeated and we must have $\sigma(2) = 2$. Similarly, $\sigma(3)$ must be less than or equal to 3 and different from the values 1 and 2 that have been used already, so $\sigma(3) = 3$. Continuing in this way we get $\sigma(i) = i$ for all i, so σ is the identity permutation.

We conclude that the only potentially nonzero term in the determinant is the one corresponding to the identity permutation, which is just the product of the diagonal entries. □

Proposition B.12. *The determinants of elementary matrices are* $\det(D_p(\lambda)) = \lambda$ *and* $\det(E_{pq}(\mu)) = 1$ *and* $\det(F_{pq}) = -1$.

Proof. The matrix $D_p(\lambda)$ is diagonal, with one of the diagonal entries being λ, and all the others being one. Proposition B.10 therefore gives $\det(D_p(\lambda)) = \lambda$. Next, the matrix $E_{pq}(\mu)$ is either upper triangular (if $p < q$) or lower triangular (if $p > q$) and all the diagonal entries are equal to one so $\det(E_{pq}(\mu)) = 1$ by Proposition B.11. Finally, consider the matrix F_{pq}, where $1 \leq p < q \leq n$. Let τ be the transposition that exchanges p and q as in Example B.6. From the definitions we see that $(F_{pq})_{ij}$ is zero when $j \neq \tau(i)$ and one when $j = \tau(i)$. It follows that the only nonzero term in $\det(F_{pq})$ is the one corresponding to τ, and that term is equal to $\operatorname{sgn}(\tau) = -1$. □

Now suppose we have permutations $\sigma\colon N \to N$ and $\tau\colon N \to N$. We then have a composite function $\sigma\circ\tau\colon N \to N$ (given by $(\sigma\circ\tau)(i) = \sigma(\tau(i))$), which is easily seen to be another permutation. The following fact is crucial for understanding the properties of the determinant:

Proposition B.13. *For σ and τ as above we have* $\operatorname{sgn}(\sigma\circ\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$.

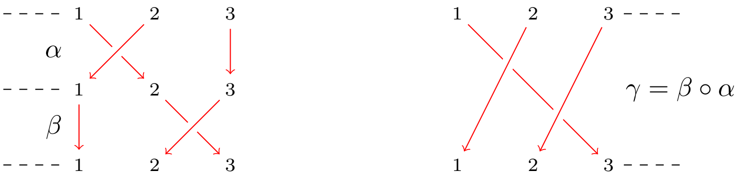
Example B.14. Take $n = 3$ and consider the permutations $\alpha, \beta\colon N \to N$ given by

$$\begin{aligned} &\alpha(1) = 2 \qquad && \alpha(2) = 1 \qquad && \alpha(3) = 3 \\ &\beta(1) = 1 && \beta(2) = 3 && \beta(3) = 2. \end{aligned}$$

Now put $\gamma = \beta\circ\alpha$. We have

$$\begin{aligned} \gamma(1) &= \beta(\alpha(1)) = \beta(2) = 3 \\ \gamma(2) &= \beta(\alpha(2)) = \beta(1) = 1 \\ \gamma(3) &= \beta(\alpha(3)) = \beta(3) = 2. \end{aligned}$$

The picture for γ is obtained by connecting the picture for β underneath the picture for α and straightening it out:



By counting crossings we see that $\operatorname{sgn}(\alpha) = \operatorname{sgn}(\beta) = -1$ but $\operatorname{sgn}(\gamma) = +1$, so $\operatorname{sgn}(\gamma) = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)$ as expected.

Proof of Proposition B.13. Consider a pair (i, j) with $1 \leq i < j \leq n$. There are four possibilities:

(a) $\tau(i) < \tau(j)$ and $\sigma(\tau(i)) < \sigma(\tau(j))$

(b) $\tau(i) < \tau(j)$ and $\sigma(\tau(i)) > \sigma(\tau(j))$
(c) $\tau(i) > \tau(j)$ and $\sigma(\tau(i)) < \sigma(\tau(j))$
(d) $\tau(i) > \tau(j)$ and $\sigma(\tau(i)) > \sigma(\tau(j))$

We let a denote the number of pairs of type (a) and so on. Recall that $l(\tau)$ is the number of pairs (i,j) as above where $\tau(i) > \tau(j)$; it is thus clear that $l(\tau) = c + d$. On the other hand, $l(\sigma \circ \tau)$ is the number of pairs (i,j) as above where $\sigma(\tau(i)) > \sigma(\tau(j))$; it is thus clear that $l(\sigma \circ \tau) = b + d$. Next, $l(\sigma)$ is the number of pairs (p,q) where $1 \leq p < q \leq n$ and $\sigma(p) > \sigma(q)$. Consider such a pair.

(b') Suppose that $\tau^{-1}(p) < \tau^{-1}(q)$. We then write $i = \tau^{-1}(p)$ and $j = \tau^{-1}(q)$. This gives a pair with $i < j$ and $\tau(i) = p < q = \tau(j)$ and $\sigma(\tau(i)) = \sigma(p) > \sigma(q) = \sigma(\tau(j))$. Thus, the pair (i,j) is an instance of case (b) above, and it is not hard to see that every instance of case (b) arises in this way, precisely once.

(c') Suppose instead that $\tau^{-1}(p) > \tau^{-1}(q)$. We then write $i = \tau^{-1}(q)$ and $j = \tau^{-1}(p)$. This gives a pair with $i < j$ and $\tau(i) = q > p = \tau(j)$ and $\sigma(\tau(i)) = \sigma(q) < \sigma(p) = \sigma(\tau(j))$. Thus, the pair (i,j) is an instance of case (c) above, and it is not hard to see that every instance of case (c) arises in this way, precisely once.

From this analysis, we see that $l(\sigma) = b + c$. This gives

$$l(\sigma) + l(\tau) = (b + c) + (c + d) = l(\sigma \circ \tau) + 2d$$

and thus

$$\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) = (-1)^{l(\sigma)+l(\tau)} = (-1)^{l(\sigma\circ\tau)+2d} = (-1)^{l(\sigma\circ\tau)} = \operatorname{sgn}(\sigma \circ \tau).$$

□

Corollary B.15. *For any permutation σ we have* $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$.

Proof. The composite $\sigma \circ \sigma^{-1}$ is the identity permutation ι, so the Proposition gives $\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\iota) = 1$. It follows that $\operatorname{sgn}(\sigma^{-1}) = 1/\operatorname{sgn}(\sigma)$, but this is the same as $\operatorname{sgn}(\sigma)$ because $\operatorname{sgn}(\sigma) = \pm 1$. $\square$

Corollary B.16. *For any square matrix A, we have* $\det(A^T) = \det(A)$.

Proof. Let a_{ij} be the entry in the i'th row of the j'th column of A, and let b_{ij} be the entry in the i'th row of the j'th column of A^T. From the definition of A^T we just have $b_{ij} = a_{ji}$. This gives

$$\det(A^T) = \sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)} =$$

$$\sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}.$$

We will rewrite this in terms of the index $j = \sigma(i)$, so $i = \sigma^{-1}(j)$. As i runs from 1 to n we see that j also runs from 1 to n in a different order, but it is harmless to write the terms of a product in a different order, so we have

$$\det(A^T) = \sum_{\text{permutations } \sigma} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j,\sigma^{-1}(j)}.$$

We now write $\tau = \sigma^{-1}$. Every permutation is the inverse of precisely one other permutation, so taking the sum over all possible σ's is the same as taking the sum over all possible τ's. Moreover, Corollary B.16 tells us that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)$. We thus have

$$\det(A^T) = \sum_{\text{permutations } \tau} \operatorname{sgn}(\tau) \prod_{j=1}^{n} a_{j,\tau(j)}.$$

This is visibly the same as the definition of $\det(A)$ (except that two dummy indices have been renamed, but that makes no difference). $\square$

The single most important fact about determinants is as follows.

Theorem B.17. *If A and B are $n\times n$ matrices, then* $\det(AB) = \det(A)\det(B)$.

Before giving the proof, we will have some preliminary discussion about expanding products of sums. It is a long but straightforward calculation to see that

$$\begin{aligned}
&(p_1+p_2+p_3)(q_1+q_2+q_3)(r_1+r_2+r_3)\\
=&p_1q_1r_1+p_1q_1r_2+p_1q_1r_3+\\
&p_1q_2r_1+p_1q_2r_2+p_1q_2r_3+\\
&p_1q_3r_1+p_1q_3r_2+p_1q_3r_3+\\
&p_2q_1r_1+p_2q_1r_2+p_2q_1r_3+\\
&p_2q_2r_1+p_2q_2r_2+p_2q_2r_3+\\
&p_2q_3r_1+p_2q_3r_2+p_2q_3r_3+\\
&p_3q_1r_1+p_3q_1r_2+p_3q_1r_3+\\
&p_3q_2r_1+p_3q_2r_2+p_3q_2r_3+\\
&p_3q_3r_1+p_3q_3r_2+p_3q_3r_3.
\end{aligned}$$

To write this in a more condensed way, note that each of the 27 terms arises by choosing one of the three terms p_i from the first bracket, one of the three terms q_j from the second bracket and one of the three terms r_k from the third bracket, and multiplying them together. This gives

$$(p_1+p_2+p_3)(q_1+q_2+q_3)(r_1+r_2+r_3) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} p_iq_jr_k.$$

Now suppose we have an array of numbers m_{ij} for $1 \leq i, j \leq 3$. By just changing notation slightly in the above formula we get

$$(m_{11} + m_{12} + m_{13})(m_{21} + m_{22} + m_{23})(m_{31} + m_{32} + m_{33}) =$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} m_{1i} m_{2j} m_{3k}.$$

The left hand side can be rewritten as

$$\left(\sum_{j=1}^{3} m_{1j} \right) \left(\sum_{j=1}^{3} m_{2j} \right) \left(\sum_{j=1}^{3} m_{3j} \right)$$

or as $\prod_{i=1}^{3} \sum_{j=1}^{3} m_{ij}$. We now consider the right hand side. Write $N = \{1, 2, 3\}$ as before. For any three numbers i, j and k as above we can define a function $\theta \colon N \to N$ by $\theta(1) = i$ and $\theta(2) = j$ and $\theta(3) = k$. Taking the sum over all possible i, j and k is the same as taking the sum over all possible functions $\theta \colon N \to N$. The product $m_{1i} m_{2j} m_{3k}$ can be rewritten as $\prod_{u=1}^{3} m_{u,\theta(u)}$. This proves the following fact:

Lemma B.18. *For any system of numbers m_{ij} (where $1 \leq i, j \leq n$) we have*

$$\prod_{i=1}^{n} \sum_{j=1}^{n} m_{ij} = \sum_{\textit{functions } \theta} \prod_{i=1}^{n} m_{i,\theta(i)}.$$

(Here θ runs over all functions from the set $N = \{1, 2, \ldots, n\}$ to itself.)

(We have really only discussed the case $n = 3$, but it should be clear that it works in general.)

Proof of Theorem B.17. Put $C = AB$. It is straightforward to check that the entries in C are given by $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$. For example, the $n = 3$ case is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32} \end{bmatrix}$$

We thus have

$$\det(C) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} c_{i,\sigma(i)} = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{j,\sigma(i)}.$$

Here we have a sum of products, so it can be expanded out as in Lemma B.18 (with $m_{ij} = a_{ij} b_{j,\sigma(i)}$). This gives

$$\det(C) = \sum_{\sigma} \sum_{\theta} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\theta(i)} b_{\theta(i),\sigma(i)}.$$

In this sum, θ is a function $N \to N$ which might or might not be a permutation. We write Δ for the sum of all the terms where θ is a permutation, and Δ' for the sum of the remaining terms where θ is not a permutation, so $\det(C) = \Delta + \Delta'$. We will show that $\Delta = \det(A)\det(B)$ and $\Delta' = 0$, which clearly implies the theorem.

When θ is a permutation we can write $\phi = \sigma \circ \theta^{-1}$, so $\sigma = \phi \circ \theta$. As σ determines ϕ and *vice versa*, taking the sum over all σ is the same as taking the sum over all ϕ. Note also that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\theta)\operatorname{sgn}(\phi)$ by Proposition B.13. This gives

$$\Delta = \sum_{\theta,\phi} \operatorname{sgn}(\theta)\operatorname{sgn}(\phi) \prod_{i=1}^{n} a_{i,\theta(i)} b_{\theta(i),\phi(\theta(i))} =$$

$$\sum_{\theta,\phi}\left(\operatorname{sgn}(\theta)\prod_{i=1}^{n}a_{i,\theta(i)}\right)\left(\operatorname{sgn}(\phi)\prod_{i=1}^{n}b_{\theta(i),\phi(\theta(i))}\right).$$

Note that as i runs from 1 to n, the index $j = \theta(i)$ also runs from 1 to n (probably in a different order). We can use this to rewrite the last term above as a product over j instead of a product over i. This gives

$$\Delta = \left(\sum_{\theta}\operatorname{sgn}(\theta)\prod_{i=1}^{n}a_{i,\theta(i)}\right)\left(\sum_{\phi}\operatorname{sgn}(\phi)\prod_{j=1}^{n}b_{j,\phi(j)}\right) =$$

$$\det(A)\det(B)$$

as claimed.

Now consider a function $\theta\colon N \to N$ that is not a permutation. As θ is not a permutation, it must send at least two different indices to the same place. We can thus choose p and q with $p < q$ but $\theta(p) = \theta(q)$. Let τ be the transposition that exchanges p and q (as in Example B.6) and recall that $\tau = \tau^{-1}$ and $\operatorname{sgn}(\tau) = -1$. Because $\theta(p) = \theta(q)$ we have $\theta\circ\tau = \theta$. Next, for any permutation σ we put

$$\Gamma(\theta,\sigma) = \operatorname{sgn}(\sigma)\prod_{i=1}^{n}b_{\theta(i),\sigma(i)}.$$

We claim that $\Gamma(\theta,\sigma) = -\Gamma(\theta,\sigma\circ\tau)$. Indeed, we have

$$\Gamma(\theta,\sigma\circ\tau) = \operatorname{sgn}(\sigma\circ\tau)\prod_{i=1}^{n}b_{\theta(i),\sigma(\tau(i))}.$$

We can rewrite the product in terms of the index $j = \tau(i)$, recalling that $\theta(j) = \theta(\tau(i)) = \theta(i)$ because $\theta(p) = \theta(q)$. We also note that $\operatorname{sgn}(\sigma\circ\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) = -\operatorname{sgn}(\sigma)$. This gives

$$\Gamma(\theta, \sigma \circ \tau) = -\operatorname{sgn}(\sigma) \prod_{j=1}^{n} b_{\theta(j),\sigma(j)} = -\Gamma(\theta, \sigma).$$

as claimed. Now consider the sum $\Gamma(\theta) = \sum_\sigma \Gamma(\theta, \sigma)$. We can divide the permutations into two groups: those for which $\sigma(p) < \sigma(q)$, and those for which $\sigma(p) > \sigma(q)$. If σ is in the first group then $\sigma \circ \tau$ is in the second group and *vice-versa*. It follows that the terms $\Gamma(\theta, \sigma)$ from the first group cancel the terms $\Gamma(\theta, \sigma)$ from the second group, leaving $\Gamma(\theta) = 0$.

Finally, from our earlier expansion of $\det(C)$ we have

$$\Delta' = \sum_\theta \left(\prod_{i=1}^{n} a_{i,\theta(i)} \right) \Gamma(\theta),$$

where the sum runs over all functions $\theta \colon N \to N$ that are not permutations. We have seen that $\Gamma(\theta) = 0$, so $\Delta' = 0$ as required. $\square$

Corollary B.19. *Suppose that A and B are $n \times n$ matrices and that A can be reduced to B by a sequence of row operations. For each step in the reduction where we multiply some row by a factor λ, we take a factor of λ. For every step where we exchange two rows, we take a factor of -1. Let μ be the product of these factors. Then $\det(A) = \det(B)/\mu$. In particular, if $B = I_n$ then $\det(A) = 1/\mu$.*

Proof. We have $B = U_r U_{r-1} \cdots U_1 A$, where the U_i are elementary matrices corresponding to the steps in the row reduction. This implies that $\det(B) = \det(A) \prod_i \det(U_i)$. If $U_i = D_p(\lambda)$ (corresponding to multiplying a row by λ) then $\det(U_i) = \lambda$. If $U_i = E_{pq}(\mu)$ then $\det(U_i) = 1$, and if $U_i = F_{pq}$ then $\det(U_i) = -1$. The claim follows easily from this. $\square$

Definition B.20. Let A be an $n \times n$ matrix, and let p and q be integers with $1 \leq p, q \leq n$.

(a) We let M_{pq} be the matrix obtained by deleting the p'th row and the q'th column from A. This is a square matrix of shape $(n-1) \times (n-1)$.
(b) We put $m_{pq} = \det(M_{pq})$.
(c) We define another number m^*_{pq} as follows. The determinant $\det(A)$ has a term $\operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$ for each permutation $\sigma \colon N \to N$. Consider only those permutations σ for which $\sigma(p) = q$. The terms corresponding to these permutations all have a_{pq} as a factor. We write m^*_{pq} for what we get after removing that factor. In symbols, we have

$$m^*_{pq} = \sum_{\sigma(p)=q} \operatorname{sgn}(\sigma) \prod_{i \neq p} a_{i,\sigma(i)}.$$

We call the matrices M_{pq} the *minor matrices* for A, and the numbers m_{pq} the *minor determinants*. The number m^*_{pq} is called the *cofactor* for a_{pq} in A.

Example B.21. Consider the case where $n = 2$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Here M_{pq} is a 1×1 matrix, which is just a number, and $m_{pq} = \det(M_{pq}) = M_{pq}$. This gives

$$m_{11} = M_{11} = d \qquad m_{12} = M_{12} = c$$
$$m_{21} = M_{21} = b \qquad m_{22} = M_{22} = a.$$

Next, we have $\det(A) = ad - bc$. To find m^*_{11} we find the term containing a and remove the a leaving $m^*_{11} = d$. Proceeding in the same way we get

$$m^*_{11} = d \qquad m^*_{12} = -c$$
$$m^*_{21} = -b \qquad m^*_{22} = a.$$

Example B.22. Consider the case where $n = 3$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

We have

$$M_{11} = \begin{bmatrix} e & f \\ h & i \end{bmatrix} \qquad M_{12} = \begin{bmatrix} d & f \\ g & i \end{bmatrix} \qquad M_{13} = \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$
$$M_{21} = \begin{bmatrix} b & c \\ h & i \end{bmatrix} \qquad M_{22} = \begin{bmatrix} a & c \\ g & i \end{bmatrix} \qquad M_{23} = \begin{bmatrix} a & b \\ g & h \end{bmatrix}$$
$$M_{31} = \begin{bmatrix} b & c \\ e & f \end{bmatrix} \qquad M_{32} = \begin{bmatrix} a & c \\ d & f \end{bmatrix} \qquad M_{33} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

and so

$$m_{11} = ei - fh \qquad m_{12} = di - fg \qquad m_{13} = dh - eg$$
$$m_{21} = bi - ch \qquad m_{22} = ai - cg \qquad m_{23} = ah - bg$$
$$m_{31} = bf - ce \qquad m_{32} = af - cd \qquad m_{33} = ae - bd$$

On the other hand, we have seen that

$$\det(A) = aei - afh - bdi + bfg + cdh - ceg$$

To find m_{11}^* we take all the terms involving a (giving $aei - afh$) and remove the a to leave $m_{11}^* = ei - fh$. The other terms m_{ij}^* can be determined in the same way:

$$m_{11}^* = ei - fh \qquad m_{12}^* = -di + fg \qquad m_{13}^* = dh - eg$$
$$m_{21}^* = -bi + ch \qquad m_{22}^* = ai - cg \qquad m_{23}^* = -ah + bg$$
$$m_{31}^* = bf - ce \qquad m_{32}^* = -af + cd \qquad m_{33}^* = ae - bd.$$

It is easy to see that in both the above examples we always have $m_{pq}^* = \pm m_{pq}$. More precisely, a closer look shows that $m_{pq}^* = (-1)^{p+q} m_{pq}$ for all p and q. We will show that this rule works in general.

Proposition B.23. *For any $n \times n$ matrix A and any numbers p, q with $1 \leq p, q \leq n$ we have $m_{pq}^* = (-1)^{p+q} m_{pq}$.*

Proof. Fix p and q, and write B for M_{pq}. The entries are given by

$$b_{ij} = \begin{cases} a_{i,j} & \text{if } i < p \text{ and } j < q \\ a_{i,j+1} & \text{if } i < p \text{ and } j \geq q \\ a_{i+1,j} & \text{if } i \geq p \text{ and } j < q \\ a_{i+1,j+1} & \text{if } i \geq p \text{ and } j \geq q. \end{cases}$$

For a more convenient way to write this, define permutations $\lambda, \rho \colon N \to N$ by

$$\lambda(i) = \begin{cases} i & \text{if } 1 \leq i < p \\ i+1 & \text{if } p \leq i < n \\ p & \text{if } i = n \end{cases}$$

$$\rho(i) = \begin{cases} i & \text{if } 1 \leq i < q \\ i+1 & \text{if } q \leq i < n \\ q & \text{if } i = n. \end{cases}$$

We then have $b_{ij} = a_{\lambda(i),\rho(j)}$ for all i, j with $1 \leq i, j \leq n-1$. This gives

$$m_{pq} = \det(B) = \sum_{\tau} \operatorname{sgn}(\tau) \prod_{i=1}^{n-1} b_{i,\tau(i)} =$$

$$\sum_{\tau} \operatorname{sgn}(\tau) \prod_{i=1}^{n-1} a_{\lambda(i),\rho(\tau(i))}.$$

Here τ runs over all permutations of $\{1, \ldots, n-1\}$. Any such permutation can be extended to a permutation τ^+ of $\{1, \ldots, n\}$ by putting $\tau^+(n) = n$. It is clear that this does not introduce any additional reversals, so $\operatorname{sgn}(\tau^+) = \operatorname{sgn}(\tau)$. We now rewrite the above expression for m_{pq} using the index $j = \lambda(i)$ instead of i. As i runs through the numbers $1, \ldots, n$ excluding n, we see that j runs through the numbers $1, \ldots, n$ excluding

$\lambda(n) = p$. We also put $\sigma = \rho\circ\tau^+\circ\lambda^{-1}$, so $\rho(\tau(i)) = \sigma(j)$. Note that $\lambda^{-1}(p) = n$ and $\tau^+(n) = n$ and $\rho(n) = q$, so $\sigma(p) = q$. Conversely, if σ is any permutation of $\{1, \dots, n\}$ with $\sigma(p) = q$ then the composite $\rho^{-1}\circ\sigma\circ\lambda$ sends n to n, so it has the form τ^+ for som permutation τ of $\{1\dots, n-1\}$. It follows that summing over all τ is the same as summing over all σ with $\sigma(p) = q$. Note also that $\operatorname{sgn}(\tau) = \operatorname{sgn}(\lambda)\operatorname{sgn}(\rho)\operatorname{sgn}(\sigma)$. We therefore have

$$m_{pq} = \sum_{\sigma(p)=q} \operatorname{sgn}(\lambda)\operatorname{sgn}(\rho)\operatorname{sgn}(\sigma)\prod_{j\neq p} a_{j,\sigma(j)} = \operatorname{sgn}(\lambda)\operatorname{sgn}(\rho)m^*_{pq}.$$

Next, the reversals for λ are just the pairs (i, n) with $p \leq i < n$, so $l(\lambda) = n - p$ and $\operatorname{sgn}(\lambda) = (-1)^{n-p}$. Similarly we have $\operatorname{sgn}(\rho) = (-1)^{n-q}$ and so $\operatorname{sgn}(\lambda)\operatorname{sgn}(\rho) = (-1)^{2n-p-q} = (-1)^{p+q}$. We therefore have $m_{pq} = (-1)^{p+q}m^*_{pq}$ as claimed. □

We can now show that determinants can be "expanded along the top row".

Proposition B.24. *Let A be an $n \times n$ matrix. Then*

$$\det(A) = \sum_{q=1}^{n}(-1)^{q+1}a_{1q}m_{1q} = a_{11}m_{11} - a_{12}m_{12} + \cdots \pm a_{1n}m_{1n}.$$

In the case $n = 3$, this is just the traditional rule

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Proof. By definition we have

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

If we consider only permutations σ that satisfy $\sigma(1) = q$, then the sum of the corresponding terms is

$$\sum_{\sigma(1)=q} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = a_{1q} \sum_{\sigma(1)=q} \operatorname{sgn}(\sigma) \prod_{i\neq 1} a_{i,\sigma(i)} = a_{1q} m^*_{1q}.$$

Taking the sum over all q, we get $\det(A) = \sum_{q=1}^{n} a_{1q} m^*_{1q}$. On the other hand, Proposition B.23 tells us that $m^*_{1q} = (-1)^{1+q} m_{1q}$, so we get $\det(A) = \sum_{q=1}^{n} (-1)^{q+1} a_{1q} m_{1q}$ as claimed. □

Definition B.25. Let A be an $n \times n$ matrix. The *adjugate* of A is the $n \times n$ matrix $\operatorname{adj}(A)$ with entries

$$\operatorname{adj}(A)_{jk} = m^*_{kj} = (-1)^{k+j} m_{kj}.$$

Proposition B.26. *For any A we have $A \operatorname{adj}(A) = \operatorname{adj}(A) A = \det(A)\, I_n$.*

Proof. Put $B = A \operatorname{adj}(A)$, which has entries

$$b_{ik} = \sum_{j=1}^{n} a_{ij} \operatorname{adj}(A)_{jk} = \sum_{j=1}^{n} a_{ij} m^*_{kj}.$$

In particular, the diagonal entries are $b_{ii} = \sum_{j=1}^{n} a_{ij} m^*_{ij}$, which is equal to $\det(A)$ by the same argument that we used for Proposition B.24.

Now consider an off-diagonal entry b_{ik} with $i \neq k$. This can be expanded as

$$b_{ik} = \sum_{j} a_{ij} m^*_{kj} = \sum_{j} \sum_{\sigma(k)=j} \operatorname{sgn}(\sigma)\, a_{ij} \prod_{l\neq k} a_{l,\sigma(l)}.$$

As a first simplification, we can write j as $\sigma(k)$ and then we do not need to mention j any more, we just have a single sum over all permutations σ. Next, one of the terms in the product is $a_{i,\sigma(i)}$ (corresponding to $l = i$) and it will be convenient to write that separately. This leaves $b_{ik} = \sum_\sigma \Gamma(\sigma)$, where

$$\Gamma(\sigma) = \operatorname{sgn}(\sigma)\, a_{i,\sigma(k)}\, a_{i,\sigma(i)} \prod_{l \neq i,k} a_{l,\sigma(l)}.$$

Now let τ be the transposition that exchanges i and k. It is not hard to see that $\Gamma(\sigma \circ \tau) = -\Gamma(\sigma)$. Using this we see that the terms $\Gamma(\sigma)$ with $\sigma(i) < \sigma(k)$ cancel against the terms $\Gamma(\sigma)$ with $\sigma(i) > \sigma(k)$, leaving $b_{ik} = 0$. This completes the proof that $A\,\mathrm{adj}(A) = \det(A) I_n$, and a similar argument shows that we also have $\mathrm{adj}(A)A = \det(A) I_n$. □

Theorem B.27. *Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$. If so, the inverse is given by $A^{-1} = \mathrm{adj}(A)/\det(A)$.*

Proof. First suppose that A is invertible, so there is an inverse matrix A^{-1} with $AA^{-1} = I_n$. Using Theorem B.17 we deduce that $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$, and this clearly means that $\det(A)$ cannot be zero.

Conversely, suppose that $\det(A) \neq 0$. We can then divide the matrix $\mathrm{adj}(A)$ by the number $\det(A)$ to get a matrix $B = \mathrm{adj}(A)/\det(A)$. After rearranging the equations in Proposition B.26 we see that $AB = BA = I_n$, so A is invertible with inverse B. □