The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2006–2007

Linear Mathematics for Applications 2 hours

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 For this question, it is given that the matrix

$$A := \begin{pmatrix} 3 & -3 & 1 & 5 & 1 & 5 & 5 \\ 1 & -1 & 1 & 1 & 1 & 3 & -1 \\ 2 & -2 & 1 & 3 & 0 & 5 & 2 \\ 2 & -2 & 0 & 4 & 0 & 2 & 1 \end{pmatrix}$$

can be transformed by elementary row operations into the matrix

$$E := \begin{pmatrix} 1 & -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Also, let

$$v_1 = (3,1,2,2), \quad v_2 = (-3,-1,-2,-2), \quad v_3 = (1,1,1,0), \quad v_4 = (5,1,3,4),$$

$$v_5 = (1,1,0,0), \quad v_6 = (5,3,5,2), \quad v_7 = (5,-1,2,1).$$

(i) Express v_6 as a linear combination of v_1, v_3, v_4, v_5. ***(2 marks)***

(ii) Determine the general solution of the system of linear equations $AX = 0$, where $X := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{pmatrix}^T$, and write your general solution in (column) vector form. ***(9 marks)***

(iii) Show that the set $\mathcal{N}_A := \left\{ v \in \mathbb{R}^7 : Av = 0 \right\}$ is a subspace of $\mathbb{R}^7$ (where $\mathbb{R}^7$ is thought of as composed of columns). Find three vectors which span this subspace, and show that your three vectors are linearly independent. ***(5 marks)***

(iv) Let W be the subspace of $\mathbb{R}^4$ given by

$$W := \mathrm{Sp}\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}.$$

(a) Find, with justification, a basis for W that has v_5 as one member, and determine $\dim W$. ***(4 marks)***

(b) Find, with justification, a basis for W that has v_4 and v_6 as two members. ***(5 marks)***

2 Find the eigenvalues of the matrix

$$A := \begin{pmatrix} 0.98 & 0.08 \\ 0.02 & 0.92 \end{pmatrix},$$

and, for each eigenvalue, find a corresponding eigenvector. ***(6 marks)***

(i) Two sequences $u_0, u_1, u_2, \ldots, u_n, \ldots$ and $v_0, v_1, v_2, \ldots, v_n, \ldots$ of real numbers are defined inductively by the formulas $u_0 = 5$, $v_0 = 0$, and

$$\begin{aligned} u_{n+1} &= 0.98u_n + 0.08v_n, \\ v_{n+1} &= 0.02u_n + 0.92v_n, \end{aligned}$$

for all non-negative integers n. Find the exact value of u_8. ***(8 marks)***

(ii) The canteen at Little Fattenham Junior School offers (essentially) two types of lunches, namely 'the healthy option' (salads, sandwiches, fruit, yoghurt, ...) and 'fast food' (burgers, chips, pizzas, ...). Throughout Autumn 2006, the school, concerned about the increasing tendency towards obesity among the population of Britain, had provided leaflets and information for the pupils and their parents to encourage more pupils to choose the healthy option.

By December 2006, experience suggested that the school's Chief Caterer could expect that, during the Winter Term (between Christmas 2006 and Easter 2007), 8% of the pupils who had been choosing the fast food option at the beginning of a week would have changed to the healthy option by the end of that week (and the other 92% would still be choosing the fast food option), while 2% of the pupils who had been choosing the healthy option at the beginning of a week would have changed to the fast food option by the end of that week (and the other 98% would still be choosing the healthy option). With these assumptions, and with the knowledge that, at the beginning of the Winter Term on Monday 08 January 2007, precisely 90% of the pupils who regularly take lunch in the Little Fattenham Junior School's canteen were choosing the fast food option, what conclusion should the school's Chief Caterer reach about the percentage of those pupils who would be choosing the healthy option at the end of the term on Friday 23 March 2007. (You should assume that no pupil will change choice of lunch type during the half term week 12–16 February 2007 or during any weekend. Not counting the half term, the pupils have to attend school for 10 weeks during the Winter Term.) ***(11 marks)***

3 (i) For each of the following subsets L_i ($i = 1, 2, 3, 4, 5$) of $\mathbb{R}^3$, determine, with justification, whether L_i is a subspace of $\mathbb{R}^3$; in each case where L_i is a subspace of $\mathbb{R}^3$, determine $\dim L_i$.

(a) $L_1 := \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 1\}$; ***(2 marks)***

(b) $L_2 := \{(x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}$; ***(4 marks)***

(c) $L_3 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 + 3z^2 = 0\}$; ***(2 marks)***

(d) $L_4 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 + 3z^2 = -1\}$; ***(1 mark)***

(e) $L_5 := \{(x, y, z) \in \mathbb{R}^3 : x^2 - 2y^2 + 3z^2 = -1\}$. ***(2 marks)***

(ii) It is given that the matrix

$$A := \begin{pmatrix} -7 & 2 & -6 \\ -6 & 1 & -6 \\ 6 & -2 & 5 \end{pmatrix}$$

has exactly two eigenvalues, and that these are 1 and -1. For each of the following subspaces W_i ($i = 1, 2, 3, 4, 5, 6$) of $\mathbb{R}^3$ (thought of as composed of columns), determine $\dim W_i$, and justify your response.

(a) $W_1 := \{v \in \mathbb{R}^3 : (A - I_3)v = 0\}$; ***(4 marks)***

(b) $W_2 := \{v \in \mathbb{R}^3 : (A + I_3)v = 0\}$; ***(4 marks)***

(c) $W_3 := W_1 \cap W_2$; ***(1 mark)***

(d) $W_4 := \{v \in \mathbb{R}^3 : Av = 0\}$; ***(1 mark)***

(e) $W_5 = \mathrm{Sp}\{(-7, -6, 6)^T, (2, 1, -2)^T, (-6, -6, 5)^T\}$; ***(2 marks)***

(f) $W_6 = \mathrm{Sp}\{(-8, 2, -6)^T, (-6, 0, -6)^T, (6, -2, 4)^T\}$. ***(2 marks)***

4 (i) Let

$$A := \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ -3 & 3 & -2 & 2 & -5 \\ -1 & 1 & 0 & 1 & -1 \\ 2 & -2 & 0 & 0 & 6 \end{pmatrix}.$$

(a) Find an invertible matrix P such that PA is in reduced row echelon form. ***(8 marks)***

(b) Find an invertible matrix Q such that PAQ is in normal form. ***(5 marks)***

(ii) Let

$$B := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -1 & 0 & 3 \end{pmatrix}.$$

(a) Calculate the adjoint matrix $\operatorname{Adj} B$. ***(6 marks)***

(b) Calculate the matrix product $B(\operatorname{Adj} B)$. ***(2 marks)***

(c) Calculate the determinant $\det B$. ***(1 mark)***

(d) State whether or not B is invertible; if it is invertible, write down its inverse. ***(3 marks)***

5 Let

$$A := \begin{pmatrix} 3 & -2 & -1 \\ -2 & 0 & -2 \\ -1 & -2 & 3 \end{pmatrix}, \quad P := \begin{pmatrix} 1 & -1 & -2 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D := \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

It is given that P is an invertible matrix and that $P^{-1}AP = D$.

(i) Find the solution of the system of linear differential equations

$$\begin{aligned} y_1' &= 3y_1 - 2y_2 - y_3 \\ y_2' &= -2y_1 - 2y_3 \\ y_3' &= -y_1 - 2y_2 + 3y_3 \end{aligned}$$

for which $y_1(0) = y_2(0) = y_3(0) = 1$. ***(10 marks)***

(ii) Find an orthogonal 3×3 matrix S such that $S^TAS = D$. ***(9 marks)***

(iii) Let $Q(x, y, z)$ be the quadratic form given by

$$Q(x, y, z) = 3x^2 + 3z^2 - 4xy - 2xz - 4yz.$$

(a) Determine the rank and signature of the quadratic form $Q(x, y, z)$. ***(2 marks)***

(b) Determine the maximum and minimum values in the set

$$K := \left\{Q(a, b, c) : a, b, c \in \mathbb{R} \text{ and } a^2 + b^2 + c^2 = 1\right\}.$$ ***(2 marks)***

(c) Determine the nature of the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x, y, z) = 1$. ***(2 marks)***

End of Question Paper