SOM201 2007 Solution 1

(i) The columns of A are the transposes of $v_1, \ldots, v_7$. For $i = 1, \ldots, 7$, let D_i denote the ith column of E. Since $D_1 + 3D_3 - D_5 - D_6 = 0$, it follows that $v_1 + 3v_3 - v_5 - v_6 = 0$ and $v_6 = v_1 + 3v_3 - v_5$.

(ii) The augmented matrix of the system

$$A\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{pmatrix}^T = 0$$

is

$$(A|0) \sim \begin{pmatrix} 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The general solution of the system is therefore

$$x_7 = 0, \quad x_6 = \mu, \quad x_5 = \mu, \quad x_4 = \nu, \quad x_3 = \nu - 3\mu, \quad x_2 = \lambda, \quad x_1 = \lambda - 2\nu - \mu$$

where μ, ν, λ are arbitrary numbers.

We can write this general solution as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} \lambda - 2\nu - \mu \\ \lambda \\ \nu - 3\mu \\ \nu \\ \mu \\ \mu \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

where μ, ν, λ are arbitrary numbers. Set

$$f := \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad g := \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad h := \begin{pmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

(iii) It follows from part (ii), with the notation introduced there, that

$$\mathcal{N}_A := \{v \in \mathbb{R}^7 : Av = 0\} = \{\lambda f + \nu g + \mu h : \mu, \nu, \lambda \in \mathbb{R}\}$$

is equal to the set of all linear combinations of f, g and h, and is therefore the subspace $\mathrm{Sp}\{f, g, h\}$ of $\mathbb{R}^7$ spanned by f, g, h.

We now show that f, g and h are linearly independent. So let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\alpha f + \beta g + \gamma h = 0, \qquad \text{that is,} \qquad \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then comparison of $(2,1)$th, $(4,1)$th and $(6,1)$th entries of these column vectors shows that $\alpha = 0$, $\beta = 0$ and $\gamma = 0$. Hence f, g, h are linearly independent.

(iv)(a) Since D_1, D_3, D_5, D_7, being the four columns of the 4×4 identity matrix I_4, are linearly independent, it follows that v_1, v_3, v_5, v_7 are linearly independent. As they are therefore 4 linearly independent vectors in $\mathbb{R}^4$, they form a basis for $\mathbb{R}^4$, and this basis contains v_5. Since every vector in $\mathbb{R}^4$ can be expressed as a linear combination of v_1, v_3, v_5, v_7, it follows that $\mathbb{R}^4 = W$. Thus v_1, v_3, v_5, v_7 form a basis for W and $\dim W = 4$.

(b) Set

$$A' := \left(\begin{array}{c|c|c|c} v_1^T & v_4^T & v_6^T & v_7^T \end{array} \right) = \left(\begin{array}{c|c|c|c} 3 & 5 & 5 & 5 \\ 1 & 1 & 3 & -1 \\ 2 & 3 & 5 & 2 \\ 2 & 4 & 2 & 1 \end{array} \right).$$

Then the same sequence of elementary row operations that will transform A into E will transform A' into

$$E' := \left(\begin{array}{c|c|c|c} D_1 & D_4 & D_6 & D_7 \end{array} \right) = \left(\begin{array}{c|c|c|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

Therefore

$$A' \sim E' \sim \left(\begin{array}{c|c|c|c} 1 & 0 & 7 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) = I_4.$$

We therefore see, as in part (a) that v_1, v_4, v_6, v_7 are linearly independent. As they are therefore 4 linearly independent vectors in $\mathbb{R}^4$, they form a basis for $\mathbb{R}^4 = W$, and this basis contains v_4 and v_6.

SOM201 2007 Solution 2

Note that A is a stochastic matrix. The sum of the eigenvalues of A is equal to

$$\text{trace}(A) = 0.98 + 0.92 = 1.9.$$

Now A is a stochastic matrix and so has 1 as one eigenvalue: let λ_2 be the other. Then $1 + \lambda_2 = \text{trace}(A) = 1.9$. Hence $\lambda_2 = 0.9$.

(i) To find an eigenvector corresponding to the eigenvalue 1, we consider $(A - 1I_2)\begin{pmatrix} x & y \end{pmatrix}^T = 0$:

$$(A - 1I_2|0) = \left(\begin{array}{cc|c} -0.02 & 0.08 & 0 \\ 0.02 & -0.08 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

so that $w_1 := \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $1 =: \lambda_1$.

To find an eigenvector corresponding to λ_2, we consider $(A - \lambda_2 I_2)\begin{pmatrix} x & y \end{pmatrix}^T = 0$:

$$(A - \lambda_2 I_2|0) = \left(\begin{array}{cc|c} 0.98 - 0.9 & 0.08 & 0 \\ 0.02 & 0.92 - 0.9 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 0.08 & 0.08 & 0 \\ 0.02 & 0.02 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

so that $w_2 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A corresponding to λ_2.

(i) We can formulate the definition of the given sequences in matrix terminology as follows:

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0.98 & 0.08 \\ 0.02 & 0.92 \end{pmatrix}\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} = A\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} \quad \text{for } n > 0.$$

We now see that

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = A^n \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \text{for all } n \geq 1.$$

Since λ_1 and λ_2 are different eigenvalues of A, the eigenvectors w_1 and w_2 are linearly independent, and, as there are two of them, they span $\mathbb{R}^2$. We express $\begin{pmatrix} 5 & 0 \end{pmatrix}^T = w_1 + w_2$ as a linear combination of w_1 and w_2. Therefore, for all $n \geq 1$,

$$\begin{aligned} \begin{pmatrix} u_n \\ v_n \end{pmatrix} &= A^n \begin{pmatrix} 5 \\ 0 \end{pmatrix} = A^n (w_1 + w_2) \\ &= A^n w_1 + A^n w_2 = (1^n)w_1 + ((0.9)^n)w_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + ((0.9)^n)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Therefore $u_8 = 4 + (0.9)^8 = 4 + 0.43046721 = 4.43046721$.

(ii) For $k = 0, 1, \ldots$, let h_k (respectively f_k) be the proportion of those pupils who regularly use the canteen and who are choosing the healthy option (respectively fast food) at the end of week k, ignoring the half term week. Then the information given in the question yields that

$$\begin{aligned} h_{k+1} &= 0.98h_k + 0.08f_k \\ f_{k+1} &= 0.02h_k + 0.92f_k \end{aligned}$$

for all $k = 0, 1, 2, \ldots$. Thus the situation can be described by means of the difference equation

$$\begin{pmatrix} h_0 \\ f_0 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} h_{k+1} \\ f_{k+1} \end{pmatrix} = \begin{pmatrix} 0.98 & 0.08 \\ 0.02 & 0.92 \end{pmatrix}\begin{pmatrix} h_k \\ f_k \end{pmatrix} = A\begin{pmatrix} h_k \\ f_k \end{pmatrix} \quad \text{for } k \geq 0.$$

We express $\begin{pmatrix} 0.1 & 0.9 \end{pmatrix}^T$ as a linear combination of w_1 and w_2. Thus we seek scalars a, b such that $\begin{pmatrix} 0.1 & 0.9 \end{pmatrix}^T = aw_1 + bw_2$. This equation is equivalent to a system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c} 4 & 1 & 0.1 \\ 1 & -1 & 0.9 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0.9 \\ 4 & 1 & 0.1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0.9 \\ 0 & 5 & -3.5 \end{array}\right)$$
$$\sim \left(\begin{array}{cc|c} 1 & -1 & 0.9 \\ 0 & 1 & -0.7 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0.2 \\ 0 & 1 & -0.7 \end{array}\right).$$

Therefore $\begin{pmatrix} 0.1 & 0.9 \end{pmatrix}^T = 0.2w_1 - 0.7w_2$. We can now estimate h_{10} using the standard techniques for solving difference equations:

$$\begin{pmatrix} h_{10} \\ f_{10} \end{pmatrix} = A^{10} \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} = A^{10}(0.2w_1 - 0.7w_2) = 0.2A^{10}w_1 - 0.7A^{10}w_2$$
$$= 0.2(1)^{10}w_1 - 0.7(0.9)^{10}w_2 = 0.2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} - 0.7(0.9)^{10} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore an estimate for $100h_{10}$, the percentage of the pupils who regularly take lunch at the Little Fattenham Junior School's canteen who will be choosing the healthy option at the end of term on Friday 23 March 2007, is

$$100h_{10} = 80 - (70)(0.34867844) = 55.6 \text{ (correct to 1 decimal place).}$$

SOM201 2007 Solution 3

(i)(a) The subset L_1 of $\mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$ because it does not contain the zero vector $0_{\mathbb{R}^3} = (0,0,0)$.

(b) The subset L_2 of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ because it is the null space of the 1×3 row matrix $B := \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. Since the only row of B is non-zero, it is clear that the rank of B is 1. Therefore $\dim L_2 = \text{nullity}(B) = 3 - \text{rank}(B) = 3 - 1 = 2$.

(c) Since the square of a real number is never negative, $L_3 = \{(0,0,0)\}$, the zero subspace of $\mathbb{R}^3$, which has dimension 0.

(d) Since the square of a real number is never negative, $x^2 + 2y^2 + 3z^2 \geq 0$ for all $(x,y,z) \in \mathbb{R}^3$. Therefore L_4 is empty, and so is not a subspace of $\mathbb{R}^3$.

(e) The subset L_1 of $\mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$ because $v := (0, 1/\sqrt{2}, 0) \in L_5$ but $2v (= 2v + 0v) = (0, \sqrt{2}, 0) \notin L_5$ because $0^2 - 2(\sqrt{2})^2 + 3.0^2 = -4 \neq -1$.

(ii)(a) The subspace W_1 consists of all the eigenvectors of A corresponding to the eigenvalue 1, together with the zero vector of $\mathbb{R}^3$. Thus $\dim W_1 = \text{nullity}(A - I_3)$.

We find the nullity of $A - I_3$ by finding the rank of $A - I_3$ and using the Rank-Nullity Theorem. Now

$$A - I_3 = \begin{pmatrix} -8 & 2 & -6 \\ -6 & 0 & -6 \\ 6 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

a reduced row echelon matrix with 2 non-zero rows. Therefore $\text{rank}(A - I_3) = 2$ and $\dim W_1 = 3 - \text{rank}(A - I_3) = 3 - 2 = 1$.

(b) The subspace W_2 consists of all the eigenvectors of A corresponding to the eigenvalue -1, together with the zero vector of $\mathbb{R}^3$. Thus $\dim W_2 = \text{nullity}(A + I_3)$.

We find the nullity of $A + I_3$ by finding the rank of $A + I_3$ and using the Rank-Nullity Theorem. Now

$$A + I_3 = \begin{pmatrix} -6 & 2 & -6 \\ -6 & 2 & -6 \\ 6 & -2 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

a reduced row echelon matrix with 1 non-zero row. Therefore $\text{rank}(A + I_3) = 1$ and $\dim W_2 = 3 - \text{rank}(A + I_3) = 3 - 1 = 2$.

(c) Now $W_1 \cap W_2 = \{v \in \mathbb{R}^3 : Av = v \text{ and } Av = -v\}$. Therefore $W_1 \cap W_2 = \{(0,0,0)\} = \{0_{\mathbb{R}^3}\}$ is the zero subspace of $\mathbb{R}^3$, and so $\dim W_3 = 0$.

(d) The subspace W_4 is the set of transposes of the vectors in the null space of A; since 0 is not an eigenvalue of A, we see that W_4 is the zero subspace of $\mathbb{R}^3$, and so $\dim W_4 = 0$.

(e) By (d), the nullity of A is 0; therefore $\text{rank}(A) = 3$, by the Rank-Nullity Theorem. Now W_5 is the column space of A, and so its dimension is equal to the rank of A. Therefore $\dim W_5 = 3$.

(f) Similarly, W_6 is the column space of $(A - I_3)^T$, and so W_6 has dimension equal to $\text{rank}(A - I_3)^T = \text{rank}(A - I_3)$. We calculated this to be 2 in (a); therefore $\dim W_6 = 2$.

SOM201 2007 Solution 4

(i) (a) i) (a) We have

$$(A|I_4) = \left(\begin{array}{ccccc|cccc} 1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -3 & 3 & -2 & 2 & -5 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 6 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|cccc} 1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 2 & 4 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|cccc} 1 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|cccc} 1 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|cccc} 1 & -1 & 0 & 0 & 3 & -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 & 0 & 1 \end{array}\right).$$

Therefore

$$P := \begin{pmatrix} -2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & -1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{pmatrix}$$

is an invertible matrix such that

$$PA = \begin{pmatrix} 1 & -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} =: E,$$

in reduced row echelon form. (In fact any matrix P' obtained from P by multiplying its last row by a non-zero number will also be invertible and such that $P'A = E$.)

(b) For matrices B_1, B_2 of the same size, we shall write $B_1 \rightleftharpoons B_2$ to indicate that B_2 can be obtained from B_1 by a sequence of elementary column operations. We have

$$\left(\frac{E}{I_5}\right) = \left(\begin{array}{ccccc} 1 & -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightleftharpoons \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \rightleftharpoons \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) =: \left(\frac{N}{Q}\right),$$

where

$$N = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the matrix Q is invertible and $EQ = N$, in normal form. Thus we have found invertible matrices P and Q such that $PAQ = EQ = N$ is a matrix in normal form.

(ii)(a)

$$\mathrm{Adj}B = \begin{pmatrix} 6 & -6 & -4 \\ -10 & 6 & 8 \\ 2 & -2 & -4 \end{pmatrix}.$$

(b) Therefore

$$B(\mathrm{Adj}B) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 6 & -6 & -4 \\ -10 & 6 & 8 \\ 2 & -2 & -4 \end{pmatrix} = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}.$$

(c) Since $B(\mathrm{Adj}B) = (\det B)I_3$, it follows that $\det B = -8$.

(d) Since $\det B \neq 0$, the matrix B is invertible; its inverse is

$$\frac{1}{\det B}\mathrm{Adj}B = \begin{pmatrix} -\frac{3}{4} & \frac{3}{4} & \frac{1}{2} \\ \frac{5}{4} & -\frac{3}{4} & -1 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

SOM201 2007 Solution 5

(i) Write

$$\left(v_1 \,\middle|\, v_2 \,\middle|\, v_3 \right) := P = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Since $P^{-1}AP = D$, we have $AP = PD$, that is,

$$A\left(v_1 \,\middle|\, v_2 \,\middle|\, v_3 \right) = \left(v_1 \,\middle|\, v_2 \,\middle|\, v_3 \right)\begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \left(-2v_1 \,\middle|\, 4v_2 \,\middle|\, 4v_3 \right).$$

Thus $Av_1 = -2v_1$, $Av_2 = 4v_2$ and $Av_3 = 4v_3$, so that -2, 4 and 4 are eigenvalues of A with corresponding eigenvectors v_1, v_2 and v_3 respectively.

(i) Let

$$Y = Y(x) := \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T = \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) \end{pmatrix}^T$$

be the 3×1 column matrix whose entries are the functions y_1, y_2, y_3. The general solution of the given system of linear differential equations is

$$Y = c_1 e^{-2x} v_1 + c_2 e^{4x} v_2 + c_3 e^{4x} v_3, \text{ where } c_1, c_2, c_3 \text{ are arbitrary scalars.}$$

We require the solution for which $Y(0) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, and so we solve the system of linear equations

$$\begin{array}{rcrcrcl} c_1 & - & c_2 & - & 2c_3 & = & 1 \\ 2c_1 & & & + & c_3 & = & 1 \\ c_1 & + & c_2 & & & = & 1. \end{array}$$

The augmented matrix of this system is

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right).$$

Thus $c_1 = \frac{2}{3}$, $c_2 = \frac{1}{3}$, $c_3 = -\frac{1}{3}$, and the required solution is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \tfrac{2}{3} e^{-2x} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \tfrac{1}{3} e^{4x} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \tfrac{1}{3} e^{4x} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

(ii) Since A is symmetric, v_1 is orthogonal to v_2 and to v_3, because eigenvectors corresponding to different eigenvalues of A must be mutually orthogonal. However, v_2 and v_3 are not orthogonal.

Since $\{v \in \mathbb{R}^3 : (A - 4I_3)v = 0\}$ is a subspace of $\mathbb{R}^3$ that contains both v_2 and v_3, any non-zero linear combination of v_2 and v_3 will again be an eigenvector of A corresponding to the eigenvalue 4. We seek such an eigenvector that is orthogonal to v_2; that is, we seek $a, b \in \mathbb{R}$, not both zero, such that $av_2 + bv_3$ is (non-zero and) orthogonal to v_2, that is, such that $(-1)(-a - 2b) + a = 0$. If we take $a = 1$ and $b = -1$, we find that $v_3' := v_2 - v_3 = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$ is an eigenvector of A corresponding to the eigenvalue 4, and is orthogonal to v_2. Note that v_3' is automatically orthogonal to v_1, as they are eigenvectors of the symmetric matrix A corresponding to different eigenvalues.

We normalize v_1, v_2, and v_3', and put the results as columns of a matrix

$$S := \left(\frac{1}{\|v_1\|} v_1 \;\middle|\; \frac{1}{\|v_2\|} v_2 \;\middle|\; \frac{1}{\|v_3'\|} v_3' \right) = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Thus S is an orthogonal matrix such that

$$S^T AS = S^{-1}AS = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

(iii) (a) Since $Q(x,y,z) = (x,y,z)A(x,y,z)^T$, the rank of $Q(x,y,z)$ is equal to the number of non-zero eigenvalues of A (with repeated eigenvalues counted in accordance with the number of times they appear as roots of the characteristic polynomial of A), namely 3, while the signature of $Q(x,y,z)$ is equal to the number of positive eigenvalues of A minus the number of negative eigenvalues of A, namely $2 - 1 = 1$.

(c) The maximum and minimum values of K are the maximum and the minimum of the eigenvalues of A, namely 4 and -2, respectively.

(d) Since two eigenvalues of A are positive and one is negative, the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x,y,z) = 1$ is a one-sheeted hyperboloid.