The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2007–8

Linear Mathematics for Applications — 2 hours

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 For this question, it is given that the matrix

$$A := \begin{pmatrix} 0 & 1 & 2 & 0 & -2 & 0 & -2 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -2 & 1 & 4 & 1 & -1 & 0 & -6 \\ 1 & 2 & 3 & 3 & 6 & 1 & -7 \\ 1 & -1 & -3 & 1 & 6 & -1 & -1 \end{pmatrix}$$

can be transformed by elementary row operations into the matrix

$$E := \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Also, let

$$v_1 = (0,1,-2,1,1), \quad v_2 = (1,0,1,2,-1), \quad v_3 = (2,-1,4,3,-3), \quad v_4 = (0,0,1,3,1),$$

$$v_5 = (-2,1,-1,6,6), \quad v_6 = (0,0,0,1,-1), \quad v_7 = (-2,1,-6,-7,-1).$$

(i) Write down the column rank of A. ***(1 mark)***

(ii) Determine the general solution of the system of linear equations $AX = 0$, where $X := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{pmatrix}^T$, and write your general solution in (column) vector form. ***(9 marks)***

(iii) Show that the set $\mathcal{N}_A := \left\{ v \in \mathbb{R}^7 : Av = 0 \right\}$ is a subspace of $\mathbb{R}^7$ (where $\mathbb{R}^7$ is thought of as composed of columns). Find three vectors which span this subspace, and show that your three vectors are linearly independent. ***(5 marks)***

(iv) Let W be the subspace of $\mathbb{R}^5$ given by

$$W := \mathrm{Sp}\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}.$$

(a) Find, with justification, a basis for W, and determine $\dim W$. ***(4 marks)***

(b) Do v_5, v_6 and v_7 form a basis for W? Justify your response. ***(2 marks)***

(c) Do v_4, v_5, v_6 and v_7 form a basis for W? Justify your response. ***(4 marks)***

2 Let

$$A := \begin{pmatrix} 0.9 & 0.04 & 0 \\ 0.09 & 0.9 & 0 \\ 0.01 & 0.06 & 1 \end{pmatrix}.$$

(i) Show that two of the eigenvalues of A are 0.84 and 0.96, and determine a third eigenvalue of A. For each eigenvalue of A, find a corresponding eigenvector.

(11 marks)

(ii) Express the column vector $(1 \;\; 0 \;\; 0)^T$ as a linear combination of your three eigenvectors found in part (i). ***(4 marks)***

(iii) On 02 January 2008, every member of a certain sample $\mathcal{P}$ of multinational European corporations required its board members to travel to board meetings by air.

However, with increasing awareness of the contributions to climate change caused by air travel, it is expected that, over the coming years, the firms in $\mathcal{P}$ will change their instructions in this respect over time. In detail, it is expected that, by the end of each period of twelve months,

(a) of those firms in $\mathcal{P}$ that required air travel to board meetings at the beginning of the period, 90% will still be using air travel, 9% will have changed to high-speed rail travel, and 1% will have eliminated the need for any travel to board meetings by use of video-conferences;

(b) of those firms in $\mathcal{P}$ that required high-speed rail travel to board meetings at the beginning of the period, 4% will have changed back to air travel, 90% will still be using high-speed rail travel, and 6% will have eliminated the need for any travel to board meetings by use of video-conferences; and

(c) each firm in $\mathcal{P}$ that was using video-conferences for its board meetings at the beginning of the period would still be doing so at the end.

Under these assumptions, what percentage of the firms in $\mathcal{P}$ will be using video-conferences for their board meetings at 02 January 2024? ***(10 marks)***

3 (i) For each of the following subsets L_i ($i = 1, 2, 3, 4, 5$) of $\mathbb{R}^4$, determine, with justification, whether L_i is a subspace of $\mathbb{R}^4$; in each case where L_i is a subspace of $\mathbb{R}^4$, determine $\dim L_i$.

(a) $L_1 := \{(w, x, y, z) \in \mathbb{R}^4 : -x + y + z = 1\}$; ***(2 marks)***

(b) $L_2 := \{(w, x, y, z) \in \mathbb{R}^4 : -x + y + z = 0\}$; ***(3 marks)***

(c) $L_3 := \{(w, x, y, z) \in \mathbb{R}^4 : x^2 + y^2 + z^2 = 0\}$; ***(3 marks)***

(d) $L_4 := \{(w, x, y, z) \in \mathbb{R}^4 : x^2 + y^2 + z^2 = -4\}$; ***(2 marks)***

(e) $L_5 := \{(w, x, y, z) \in \mathbb{R}^4 : x^2 - y^2 + z^2 = -4\}$. ***(2 marks)***

(ii) Let

$$A := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -8 & -1 & 0 & 1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

For each of the following subspaces W_i ($i = 1, 2, 3, 4, 5$) of $\mathbb{R}^4$ (thought of as composed of columns), determine $\dim W_i$, and justify your response.

(a) $W_1 := \{v \in \mathbb{R}^4 : (A + I_4)v = 0\}$; ***(3 marks)***

(b) $W_2 := \{v \in \mathbb{R}^4 : (A - I_4)v = 0\}$; ***(3 marks)***

(c) $W_3 = \{v \in \mathbb{R}^4 : (A - 2I_4)v = 0\}$; ***(3 marks)***

(d) $W_4 := W_1 \cap W_2$; ***(2 marks)***

(e) $W_5 :=$ column space (A). ***(2 marks)***

4 (i) Let

$$A := \begin{pmatrix} 2 & 1 & -2 \\ 3 & 1 & -3 \\ 5 & -1 & -1 \end{pmatrix}.$$

(a) Calculate the adjoint matrix $\operatorname{Adj} A$. ***(9 marks)***

(b) Calculate the matrix product $A(\operatorname{Adj} A)$. ***(2 marks)***

(c) Calculate the determinant $\det A$. ***(1 mark)***

(d) State whether or not A is invertible; if it is invertible, write down its inverse. ***(3 marks)***

(ii) For this part, it is given that

$$P := \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ -1 & 2 & 0 \end{pmatrix}$$

is an invertible matrix such that

$$P^{-1} \begin{pmatrix} -7 & 2 & -8 \\ -3 & 0 & -3 \\ 6 & -2 & 7 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find the solution of the system of linear differential equations

$$\begin{aligned} y_1' &= -7y_1 + 2y_2 - 8y_3 \\ y_2' &= -3y_1 - 3y_3 \\ y_3' &= 6y_1 - 2y_2 + 7y_3 \end{aligned}$$

for which $y_1(0) = y_2(0) = y_3(0) = 1$. ***(10 marks)***

5 Let $Q(x, y, z)$ be the real quadratic form given by

$$Q(x, y, z) = 3x^2 - 3y^2 - 2xz + 2yz.$$

(i) Express $Q(x, y, z)$ as a sum of squares and negatives of squares of linearly independent linear forms. You should explain why your linear forms are linearly independent. ***(7 marks)***

(ii) Determine the rank and signature of the quadratic form $Q(x, y, z)$. ***(2 marks)***

(iii) Determine the nature of the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x, y, z) = 1$. ***(2 marks)***

(iv) Let

$$A := \begin{pmatrix} 3 & 0 & -1 \\ 0 & -3 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Find an invertible 3×3 matrix S such that $S^T AS =: D$ is a diagonal matrix with diagonal entries taken from the set $\{1, -1, 0\}$. You should explain why your S is invertible, and you should exhibit your S and D clearly. ***(8 marks)***

(v) Determine the maximum and minimum values in the set

$$K := \left\{ Q(a, b, c) : a, b, c \in \mathbb{R} \text{ and } a^2 + b^2 + c^2 = 1 \right\}.$$ ***(6 marks)***

End of Question Paper