SOM201 2008 Solution 1

(i) The column rank of A (which is equal to the rank of A) is 4.
(ii) The general solution is

$$x_7 = \lambda,\ x_6 = -2\lambda,\ x_5 = \mu,\ x_4 = 2\lambda - 3\mu,\ x_3 = \nu,\ x_2 = 2\lambda + 2\mu - 2\nu,\ x_1 = -\lambda - \mu + \nu,$$

where λ, μ, ν are arbitrary real numbers. Thus, in (column) vector form, the set of all solutions of $AX = 0$ is

$$\left\{ \begin{pmatrix} -\lambda - \mu + \nu \\ 2\lambda + 2\mu - 2\nu \\ \nu \\ 2\lambda - 3\mu \\ \mu \\ -2\lambda \\ \lambda \end{pmatrix} \in \mathbb{R}^7 : \lambda, \mu, \nu \in \mathbb{R} \right\}.$$

(iii) $\mathcal{N}_A$ is the set of all column vectors $v \in \mathbb{R}^7$ which are solutions of $AX = 0$. By (ii),

$$\mathcal{N}_A = \{\lambda e + \mu f + \nu g : \lambda, \mu, \nu \in \mathbb{R}\},$$

where

$$e = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \quad f = \begin{pmatrix} -1 \\ 2 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus $\mathcal{N}_A = \mathrm{Sp}\{e, f, g\}$ is the span of the vectors e, f, g, and so is a subspace of $\mathbb{R}^7$. We show that e, f, g are linearly independent.

Suppose $\alpha, \beta, \gamma \in \mathbb{R}$ are such that $\alpha e + \beta f + \gamma g = 0$. Comparison of $(7,1)$th, $(5,1)$th and $(3,1)$th entries of the two sides of this equation shows that $\alpha = 0 = \beta = \gamma$. Thus e, f, g are linearly independent.
(iv) Write

$$E = \left(D_1 \,\middle|\, D_2 \,\middle|\, D_3 \,\middle|\, D_4 \,\middle|\, D_5 \,\middle|\, D_6 \,\middle|\, D_7 \right),$$

so that D_j is the jth column of E (for $j = 1, \ldots, 7$). Note that

$$A = \left(v_1^T \,\middle|\, v_2^T \,\middle|\, v_3^T \,\middle|\, v_4^T \,\middle|\, v_5^T \,\middle|\, v_6^T \,\middle|\, v_7^T \right).$$

(a) Since D_1, D_2, D_4, D_6 (being 4 columns of I_5) are linearly independent, v_1, v_2, v_4, v_6 are linearly independent. Since

$$D_3 = -D_1 + 2D_2, \quad D_5 = D_1 - 2D_2 + 3D_4, \quad D_7 = D_1 - 2D_2 - 2D_4 + 2D_6,$$

(and the system $AX = 0$ has exactly the same set of solutions as $EX = 0$),

$$v_3 = -v_1 + 2v_2, \quad v_5 = v_1 - 2v_2 + 3v_4, \quad v_7 = v_1 - 2v_2 - 2v_4 + 2v_6.$$

Thus v_3, v_5, v_7 are linear combinations of v_1, v_2, v_4, v_6. Therefore

$$W = \mathrm{Sp}\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} = \mathrm{Sp}\{v_1, v_2, v_4, v_6\},$$

so that v_1, v_2, v_4, v_6 form a basis for W and $\dim W = 4$.

(b) The three vectors v_5, v_6, v_7 cannot form a basis for the 4-dimensional subspace W of $\mathbb{R}^5$, because every basis for W must have 4 members.

(c) The sequence of EROs which shows that $A \sim E$ will also show, on deletion of 1st, 2nd and 3rd columns throughout, that

$$A' := \left(\begin{array}{c|c|c|c} v_4^T & v_5^T & v_6^T & v_7^T \end{array} \right) \sim \left(\begin{array}{c|c|c|c} D_4 & D_5 & D_6 & D_7 \end{array} \right).$$

Therefore

$$A' \sim \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & -2 & 0 & -2 \\ 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =: G = \left(\begin{array}{c|c|c|c} H_4 & H_5 & H_6 & H_7 \end{array} \right).$$

Since $H_7 = -5H_4 + H_5 + 2H_6$, it follows that $v_7 = -5v_4 + v_5 + 2v_6$ and $(-5)v_4 + v_5 + 2v_6 - v_7 = 0$. This shows that v_4, v_5, v_6, v_7 are linearly dependent; therefore, they cannot form a basis for W.

SOM201 2008 Solution 2

(i) We show that 0.96 is an eigenvalue of A by showing that the system of linear equations

$$(A-(0.96)I_3)(x\ y\ z)^T = 0$$

has a non-trivial solution. This system has augmented matrix

$$\left(\ A-(0.96)I_3 \mid 0\ \right) = \left(\begin{array}{ccc|c} -0.06 & 0.04 & 0 & 0 \\ 0.09 & -0.06 & 0 & 0 \\ 0.01 & 0.06 & 0.04 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 1 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 6 & 4 & 0 \\ 0 & -20 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that the system of linear equations $(A-(0.96)I_3)(x\ y\ z)^T = 0$ has a non-trivial solution, so that 0.96 is an eigenvalue of A. In particular, we see that $v_1 := (-2\ \ -3\ \ 5)^T$ is an eigenvector of A corresponding to the eigenvalue 0.96.

We show that 0.84 is an eigenvalue of A by showing that the system of linear equations

$$(A-(0.84)I_3)(x\ y\ z)^T = 0$$

has a non-trivial solution. This system has augmented matrix

$$\left(\ A-(0.84)I_3 \mid 0\ \right) = \left(\begin{array}{ccc|c} 0.06 & 0.04 & 0 & 0 \\ 0.09 & 0.06 & 0 & 0 \\ 0.01 & 0.06 & 0.16 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 6 & 16 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 6 & 16 & 0 \\ 0 & -16 & -48 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that the system of linear equations $(A-(0.84)I_3)(x\ y\ z)^T = 0$ has a non-trivial solution, so that 0.84 is an eigenvalue of A. In particular, we see that $v_2 := (2\ \ -3\ \ 1)^T$ is an eigenvector of A corresponding to the eigenvalue 0.84.

Since A is a stochastic matrix, 1 must be an eigenvalue of A. An eigenvector of A corresponding to the eigenvalue 1 is a non-zero (column) vector $v \in \mathbb{R}^3$ such that $(A-1I_3)v = 0$, and since

$$A - 1I_3 = \begin{pmatrix} -0.1 & 0.04 & 0 \\ 0.09 & -0.1 & 0 \\ 0.01 & 0.06 & 0 \end{pmatrix},$$

it is clear that $v_3 := (0\ \ 0\ \ 1)^T$ is an eigenvector of A corresponding to the eigenvalue 1.

(ii) (Since v_1, v_2, v_3 are eigenvectors of A corresponding to distinct eigenvalues of A, they are linearly independent, and so, as there are 3 of them, they form a basis for $\mathbb{R}^3$.) We find $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that $\sum_{i=1}^{3} \mu_i v_i = (1\ \ 0\ \ 0)^T$ by solving the system of linear equations given, in matrix form, by

$$\left(\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array}\right) \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix of this system

$$\left(\begin{array}{ccc|c} -2 & 2 & 0 & 1 \\ -3 & -3 & 0 & 0 \\ 5 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 5 & 0 & 1 \\ -3 & -3 & 0 & 0 \\ 5 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 5 & 0 & 1 \\ 0 & 12 & 0 & 3 \\ 0 & -24 & 1 & -5 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Hence $(1 \;\; 0 \;\; 0)^T = -\frac{1}{4}v_1 + \frac{1}{4}v_2 + 1v_3$.

(iii) For each non-negative integer n, let $w_n = (w_{n1} \;\; w_{n2} \;\; w_{n3})^T \in \mathbb{R}^3$, where $100w_{ni}$, for $i = 1, 2, 3$ is the percentage of the firms in $\mathcal{P}$ which, on 02 January $(2008 + n)$ are using air travel, high-speed train travel, and video-conferences respectively for their board meetings. The information given in the question yields that $w_{n+1} = Aw_n$ for all $n \in \mathbb{N}_0$, and that $w_0 = (1 \;\; 0 \;\; 0)^T$. It therefore follows from the standard theory of difference equations that, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} w_n &= A^n w_0 = A^n \left(-\tfrac{1}{4}v_1 + \tfrac{1}{4}v_2 + 1v_3\right) = -\tfrac{1}{4}A^n v_1 + \tfrac{1}{4}A^n v_2 + 1A^n v_3 \\ &= -\tfrac{1}{4}(0.96)^n v_1 + \tfrac{1}{4}(0.84)^n v_2 + 1^n v_3. \end{aligned}$$

To determine the expected situation on 02 January 2024, we take $n = 16$. The expected percentage of the firms in $\mathcal{P}$ that will be using video-conferences for their board meetings at 02 January 2024 is

$$\begin{aligned} 100w_{16,3} &= 100\left(-0.25 \times (0.96)^{16} \times 5 + 0.25 \times (0.84)^{16} + 1\right) \\ &\approx 100\left(-0.25 \times 0.5204 \times 5 + 0.25 \times 0.0614 + 1\right) \approx 100\left(-0.6505 + 0.0154 + 1\right) \approx 36.5 \end{aligned}$$

approximately.

SOM201 2008 Solution 3

(i)(a) The subset L_1 of $\mathbb{R}^4$ is not a subspace of $\mathbb{R}^4$ because it does not contain the zero vector $0_{\mathbb{R}^4} = (0,0,0,0)$.

(b) The subset L_2 of $\mathbb{R}^4$ is a subspace of $\mathbb{R}^4$ because it is the null space of the 1×4 row matrix $B := \begin{pmatrix} 0 & -1 & 1 & 1 \end{pmatrix}$. Since the only row of B is non-zero, it is clear that the rank of B is 1. Therefore $\dim L_2 = \text{nullity}(B) = 4 - \text{rank}(B) = 4 - 1 = 3$.

(c) Since the square of a real number is never negative,

$$L_3 = \{(w,x,y,z) \in \mathbb{R}^4 : x = y = z = 0\} = \{(w,0,0,0) : w \in \mathbb{R}\} = \text{Sp}\{(1,0,0,0)\}.$$

Thus L_3 is the span of the set containing just the one vector $e_1 := (1,0,0,0)$, and so is a subspace of $\mathbb{R}^4$. Since e_1 is non-zero, it forms a basis for L_3; therefore $\dim L_3 = 1$.

(d) Since the square of a real number is never negative, $x^2 + y^2 + z^2 \geq 0$ for all $(w,x,y,z) \in \mathbb{R}^4$. Therefore L_4 is empty, and so is not a subspace of $\mathbb{R}^4$.

(e) The subset L_5 of $\mathbb{R}^4$ is not a subspace of $\mathbb{R}^4$ because $v := (1,0,2,0) \in L_5$ but $2v (= 2v + 0v) = (2,0,4,0) \notin L_5$ because $0^2 - 4^2 + 0^2 = -16 \neq -4$.

(ii)(a) Note that $\dim W_1 = \text{nullity}(A + I_4)$. We find the nullity of $A + I_4$ by finding the rank of $A + I_4$ and using the Rank-Nullity Theorem. Now

$$A + I_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ -8 & -1 & 1 & 1 \\ 0 & -1 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -2 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

a reduced row echelon matrix with 3 non-zero rows. Therefore $\text{rank}(A + I_4) = 3$ and $\dim W_1 = 4 - \text{rank}(A + I_4) = 4 - 3 = 1$.

(b) Note that $\dim W_2 = \text{nullity}(A - I_4)$. We find the nullity of $A - I_4$ by finding the rank of $A - I_4$ and using the Rank-Nullity Theorem. Now

$$A - I_4 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -8 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a reduced row echelon matrix with 2 non-zero rows. Therefore $\text{rank}(A - I_4) = 2$ and $\dim W_2 = 4 - \text{rank}(A - I_4) = 4 - 2 = 2$.

(c) Clearly $0_{\mathbb{R}^4} \in W_3$. Furthermore, if there exists $0 \neq v \in W_3$, then $Av = 2v$, so that 2 would be an eigenvalue of A with v as a corresponding eigenvector. However, 2 is not an eigenvalue of A, since the characteristic polynomial of A,

$$\chi_A(t) = \begin{vmatrix} -1-t & 0 & 0 & 0 \\ 4 & 1-t & 0 & 0 \\ -8 & -1 & -t & 1 \\ 0 & -1 & -1 & 2-t \end{vmatrix} = -(1+t)(1-t)\left(-t(2-t)+1\right)$$
$$= -(1+t)(1-t)(t^2 - 2t + 1) = -(1+t)(1-t)^3$$

does not have 2 as a root. Therefore W_3 is the zero subspace of $\mathbb{R}^4$, and so $\dim W_3 = 0$.

(d) Now $W_1 \cap W_2 = \{v \in \mathbb{R}^4 : Av = -v \text{ and } Av = v\}$. Therefore $W_1 \cap W_2 = \{(0, 0, 0, 0)\} = \{0_{\mathbb{R}^4}\}$ is the zero subspace of $\mathbb{R}^4$, and so $\dim W_4 = 0$.

(e) The dimension of W_5 is the rank of A. Since 0 is not an eigenvalue of A (by part (c)), we must have $\{v \in \mathbb{R}^4 : Av = 0\} = \{0_{\mathbb{R}^4}\}$; hence $\text{nullity}(A) = 0$, so that $\dim W_5 = \text{rank}(A) = 4 - 0 = 4$ by the Rank-Nullity Theorem.

SOM201 2008 Solution 4

(i)(a)

$$\mathrm{Adj}A = \begin{pmatrix} -4 & 3 & -1 \\ -12 & 8 & 0 \\ -8 & 7 & -1 \end{pmatrix}.$$

(b) Therefore

$$A(\mathrm{Adj}A) = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 1 & -3 \\ 5 & -1 & -1 \end{pmatrix} \begin{pmatrix} -4 & 3 & -1 \\ -12 & 8 & 0 \\ -8 & 7 & -1 \end{pmatrix} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

(c) Since $A(\mathrm{Adj}A) = (\det A)I_3$, it follows that $\det A = -4$.

(d) Since $\det A \neq 0$, the matrix A is invertible; its inverse is

$$\frac{1}{\det A}\mathrm{Adj}A = \begin{pmatrix} 1 & -\frac{3}{4} & \frac{1}{4} \\ 3 & -2 & 0 \\ 2 & -\frac{7}{4} & \frac{1}{4} \end{pmatrix}.$$

(ii) For $i = 1, 2, 3$, let v_i denote the ith column of P. Set

$$B := \begin{pmatrix} -7 & 2 & -8 \\ -3 & 0 & -3 \\ 6 & -2 & 7 \end{pmatrix} \qquad \text{and} \qquad D := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since $P^{-1}BP = D$, we have $BP = PD$, that is,

$$B \left(\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array} \right) = \left(\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \left(\begin{array}{c|c|c} 1v_1 & 0v_2 & (-1)v_3 \end{array} \right).$$

Thus $Bv_1 = 1v_1$, $Bv_2 = 0v_2$ and $Bv_3 = (-1)v_3$, so that 1, 0 and -1 are eigenvalues of B with corresponding eigenvectors v_1, v_2 and v_3 respectively.

Let

$$Y = Y(x) := \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T = \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) \end{pmatrix}^T$$

be the 3×1 column matrix whose entries are the functions y_1, y_2, y_3. The general solution of the given system of linear differential equations is

$$Y = c_1 e^{x} v_1 + c_2 e^{0x} v_2 + c_3 e^{-x} v_3, \text{ where } c_1, c_2, c_3 \text{ are arbitrary scalars.}$$

We require the solution for which $Y(0) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, and so we solve the system of linear equations

$$\begin{array}{rcrcrcl} c_1 & - & 2c_2 & + & c_3 & = & 1 \\ & & c_2 & + & 3c_3 & = & 1 \\ -c_1 & + & 2c_2 & & & = & 1 \end{array}.$$

The augmented matrix of this system is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ -1 & 2 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 7 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array} \right).$$

Thus $c_1 = -11$, $c_2 = -5$, $c_3 = 2$, and the required solution is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = -11e^{x} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - 5 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + 2e^{-x} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}.$$

SOM201 2008 Solution 5

(i)

$$\begin{aligned} Q(x,y,z) &= 3x^2 - 3y^2 - 2xz + 2yz = 3x^2 - 2xz - 3y^2 + 2yz \\ &= \left(\sqrt{3}x - \tfrac{1}{\sqrt{3}}z\right)^2 - \left(3y^2 - 2yz + \tfrac{1}{3}z^2\right) = \left(\sqrt{3}x - \tfrac{1}{\sqrt{3}}z\right)^2 - \left(\sqrt{3}y - \tfrac{1}{\sqrt{3}}z\right)^2. \end{aligned}$$

The matrix

$$P := \begin{pmatrix} \sqrt{3} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & 1 \end{pmatrix}$$

is invertible (since it has non-zero determinant), and so

$$\begin{array}{rcrrr} x' & = & \sqrt{3}x & & - \frac{1}{\sqrt{3}}z \\ y' & = & & \sqrt{3}y & - \frac{1}{\sqrt{3}}z \\ z' & = & & & z \end{array}$$

are three linearly independent linear forms. Therefore

$$Q(x,y,z) = 3x^2 - 3y^2 - 2xz + 2yz = x'^2 - y'^2$$

where x', y' are linearly independent linear forms.

(ii) The rank of $Q(x,y,z)$ is therefore $1+1=2$ and the signature of $Q(x,y,z)$ is $1-1=0$.

(iii) Notice that $Q(x,y,z) = (x,y,z)A(x,y,z)^T$, where A is as in part (iv) of the question. As A is a real symmetric matrix, its eigenvalues are all real, and the conclusions of part (ii) mean that A must have one positive eigenvalue, one negative eigenvalue, and 0 as its third eigenvalue. Therefore the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x,y,z) = 1$ is a cylinder with an hyperbola as base.

(iv) Since

$$\begin{aligned} Q(x,y,z) &= 3x^2 - 3y^2 - 2xz + 2yz = (x,y,z)A(x,y,z)^T \\ &= (x,y,z)P^T(P^{-1})^T A P^{-1} P (x,y,z)^T \\ &= (x',y',z')(P^{-1})^T A P^{-1}(x',y',z')^T = (x',y',z') \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \end{aligned}$$

and $(P^{-1})^T A P^{-1}$ is symmetric, we see that $S := P^{-1}$ is a 3×3 matrix (that is invertible because it is the inverse of an invertible matrix) such that $S^T A S$ is a diagonal matrix with entries $1, -1$ and 0 along its diagonal. We find P^{-1}:

$$\begin{aligned} (P|I_3) &= \left(\begin{array}{ccc|ccc} \sqrt{3} & 0 & -\frac{1}{\sqrt{3}} & 1 & 0 & 0 \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right). \end{aligned}$$

Therefore

$$S = P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{3} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

is an invertible matrix such that

$$S^T AS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = D.$$

(v) We find the eigenvalues of A. They are the roots of the characteristic polynomial $\chi_A(t)$ of A. Now

$$\chi_A(t) = \begin{vmatrix} 3-t & 0 & -1 \\ 0 & -3-t & 1 \\ -1 & 1 & -t \end{vmatrix} = (3-t)\,(t(3+t)-1) + 3 + t = -t^3 + 11t$$
$$= -t(\sqrt{11}-t)(\sqrt{11}+t).$$

Therefore the eigenvalues of A are $\sqrt{11}$, 0 and $-\sqrt{11}$.

The maximum and minimum values of K are the maximum and the minimum of the eigenvalues of A, namely $\sqrt{11}$ and $-\sqrt{11}$, respectively.