The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2008–2009

Linear Mathematics for Applications 2 hours

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 For this question, it is given that the matrix

$$A := \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 2 & 0 & 2 & 1 & 3 \\ 0 & 1 & 2 & 1 & 1 & 2 & 2 \\ 1 & 0 & -1 & 0 & 0 & 1 & 5 \end{pmatrix}$$

can be transformed by elementary row operations into the matrix

$$E := \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Write down the rank of A. ***(1 mark)***

(ii) Determine the general solution of the system of linear equations $AX = 0$, where $X := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{pmatrix}^T$, and write your general solution in (column) vector form. ***(9 marks)***

(iii) Show that the set $\mathcal{N}_A := \left\{ v \in \mathbb{R}^7 : Av = 0 \right\}$ is a subspace of $\mathbb{R}^7$. State clearly, without proof, any results from the lectures which you use.

Find three vectors which span this subspace, and show that your three vectors are linearly independent. ***(5 marks)***

(iv) Let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be the columns of the above matrix A. Let W be the subspace of $\mathbb{R}^5$ given by

$$W := \mathrm{Sp}\left\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\right\}.$$

(a) What is the dimension of W ? ***(2 marks)***

(b) Find a basis of W with v_4 as a member. ***(3 marks)***

(c) Find a basis of W with v_6 and v_7 as members. ***(3 marks)***

(d) Is the set $\{v_1, v_2, v_3, v_5\}$ a basis for W? Justify your response. ***(2 marks)***

2 Let

$$A := \begin{pmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{pmatrix}.$$

(i) Find the eigenvalues of A, and for each eigenvalue a corresponding eigenvector. ***(11 marks)***

(ii) Express the column vector $(0.3 \;\; 0.7)^T$ as a linear combination of your two eigenvectors found in part (i). ***(4 marks)***

(iii) A particular airline, Sheffield Air, has been analysing switching of flights by Business Class customers on a particular route. It has found the following.

- If a customer's last flight was with Sheffield Air, the probability that the next flight on the route is also with Sheffield Air is 85%.
- If a customer's last flight was with a competing airline, the probability that the next flight is with a competing airline is 90%.

Currently, Sheffield Air has 30% of the Business Class market on the stated route. Assuming an average customer makes 1 flight per year, what will Sheffield Air's share of the market be in ten years' time ? ***(10 marks)***

3 (i) For each of the following subsets L_i $(i = 1, 2, 3, 4, 5)$ of $\mathbb{R}^4$, determine, with justification, whether L_i is a subspace of $\mathbb{R}^3$. In each case where L_i is a subspace of $\mathbb{R}^4$, determine $\dim L_i$.

(a) $L_1 := \{(w, x, y, z) \in \mathbb{R}^4 : w + x + y + z = 0\}$; ***(3 marks)***

(b) $L_2 := \{(w, x, y, z) \in \mathbb{R}^4 : w + x + y + z = 1\}$; ***(2 marks)***

(c) $L_3 := \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 = 0\}$; ***(2 marks)***

(d) $L_4 := \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 = 1\}$; ***(2 marks)***

(e) $L_5 := \{(w, x, y, z) \in \mathbb{R}^4 : w^3 + x^3 + y^3 + z^3 = 0\}$. ***(3 marks)***

(ii) Let

$$A := \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

For each of the following subspaces W_i $(i = 1, 2, 3, 4, 5)$ of $\mathbb{R}^3$, determine $\dim W_i$, and justify your response.

(a) $W_1 := \{v \in \mathbb{R}^3 : (A + I_3)v = 0\}$; ***(3 marks)***

(b) $W_2 := \{v \in \mathbb{R}^4 : (A - I_3)v = 0\}$; ***(3 marks)***

(c) $W_3 = \{v \in \mathbb{R}^4 : Av = 0\}$; ***(3 marks)***

(d) $W_4 := W_1 \cap W_2$; ***(2 marks)***

(e) $W_5 :=$ column space (A). ***(2 marks)***

4 (i) Calculate the determinants of the following matrices.

(a)

$$A := \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

(2 marks)

(b)

$$B := \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(2 marks)

(c)

$$C := \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

(3 marks)

(d) $D := BC$ ***(2 marks)***

(ii) Find the ranks of each of the matrices in part (i). ***(6 marks)***

(iii) For this part, it is given that

$$P := \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}$$

is an invertible matrix such that

$$P^{-1} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Find the solution of the system of linear differential equations

$$\begin{array}{rcrcrcr} y_1' & = & y_1 & & & - & y_3 \\ y_2' & = & y_1 & + & 2y_2 & + & y_3 \\ y_3' & = & 2y_1 & + & 2y_2 & + & 3y_3 \end{array}$$

for which $y_1(0) = 1$, $y_2(0) = 1$, and $y_3(0) = 2$. ***(10 marks)***

5 (i) Let $Q(x, y, z)$ be the real quadratic form given by

$$Q(x, y, z) = x^2 + 2xy + 2y^2 - 2yz.$$

(a) Express $Q(x, y, z)$ as a sum of squares and negatives of squares of linearly independent linear forms. You should explain why your linear forms are linearly independent. ***(7 marks)***

(b) Determine the rank and signature of the quadratic form $Q(x, y, z)$. ***(2 marks)***

(c) Let

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Find an invertible 3×3 matrix S such that $S^T AS =: D$ is a diagonal matrix with diagonal entries taken from the set $\{1, -1, 0\}$. You should explain why your S is invertible, and you should exhibit your S and D clearly. ***(8 marks)***

(d) Determine the nature of the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x, y, z) = 1$. ***(2 marks)***

(ii) Let $R(x, y, z)$ be the real quadratic form given by

$$R(x, y, z) = x^2 + y^2 + z^2 - 4yz.$$

Determine the maximum and minimum values in the set

$$K := \left\{ R(a, b, c) : a, b, c \in \mathbb{R} \text{ and } a^2 + b^2 + c^2 = 1 \right\}.$$ ***(6 marks)***

End of Question Paper