SOM201 2008 Solution 1

(i) The rank of A (which is equal to the column rank of E) is 4.
(ii) The general solution is

$$x_7 = 0,\ x_6 = \lambda,\ x_5 = \mu,\ x_4 = -\lambda + \mu,\ x_3 = \nu,\ x_2 = -\lambda - 2\mu - 2\nu,\ x_1 = -\lambda + \nu,$$

where λ, μ, ν are arbitrary real numbers. Thus, in (column) vector form, the set of all solutions of $AX = 0$ is

$$\left\{ \begin{pmatrix} -\lambda + \nu \\ -\lambda - 2\mu - 2\nu \\ \nu \\ -\lambda + \mu \\ \mu \\ \lambda \\ 0 \end{pmatrix} \in \mathbb{R}^7 : \lambda, \mu, \nu \in \mathbb{R} \right\}.$$

(iii) $\mathcal{N}_A$ is the set of all column vectors $v \in \mathbb{R}^7$ which are solutions of $AX = 0$. By (ii),

$$\mathcal{N}_A = \{\lambda e + \mu f + \nu g : \lambda, \mu, \nu \in \mathbb{R}\},$$

where

$$e = \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus $\mathcal{N}_A = \mathrm{Sp}\{e, f, g\}$ is the span of the vectors e, f, g, and so is a subspace of $\mathbb{R}^7$. We show that e, f, g are linearly independent.

Suppose $\alpha, \beta, \gamma \in \mathbb{R}$ are such that $\alpha e + \beta f + \gamma g = 0$. Comparison of 6-th, 5-th and 3-rd entries of the two sides of this equation shows that $\alpha = 0 = \beta = \gamma$. Thus e, f, g are linearly independent.
(iv) Write

$$E = \left(\begin{array}{c|c|c|c|c|c|c} D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 \end{array} \right),$$

so that D_j is the jth column of E (for $j = 1, \ldots, 7$). Note that

$$A = \left(\begin{array}{c|c|c|c|c|c|c} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \right).$$

(a) The dimension of W is the rank of A, which we previously calculated to be 4.

(b) The vectors D_1, D_2, D_4, D_7 (being 4 columns of I_5) are linearly independent. So the vectors v_1, v_2, v_4, v_7 are linearly independent.

Since the space W has dimension 4, a linearly independent set of 4 vectors is a basis. Thus $\{v_1, v_2, v_4, v_7\}$ is a basis for W.

(c) The vectors D_1, D_2, D_6, D_7 are linearly independent. So the vectors v_1, v_2, v_6, v_7 are also linearly independent.

Since the space W has dimension 4, a linearly independent set of 4 vectors is a basis. Thus $\{v_1, v_2, v_6, v_7\}$ is a basis for W.

(d) The vector v_7 does not belong to the span of the vectors v_1, v_2, v_3, v_5. But $v_7 \in W$. So the set $\{v_1, v_2, v_3, v_5\}$ is not a basis for W.

Alternatively, argue that the set $\{v_1, v_2, v_3, v_5\}$ is linearly dependent.

SOM201 2008 Solution 2

(i) The matrix A is stochastic. Therefore 1 is an eigenvalue of A.

The sum of the eigenvalues of A is the trace, $\mathrm{tr}(A)$, which is $0.85 + 0.9 = 1.75$.

Hence $1.75 - 1 = 0.75$ is also an eigenvalue of A.

Let $(x, y)^T$ be an eigenvector of A with eigenvalue 1. Then

$$(A - I_2)(x, y)^T = 0$$

This system of linear equations has augmented matrix

$$\left(A - (0.96)I_3 \mid 0 \right) = \left(\begin{array}{cc|c} -0.15 & 0.1 & 0 \\ 0.15 & -0.1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} -15 & 10 & 0 \\ 0 & 0 & 0 \end{array} \right)$$
$$\sim \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

It follows that the system of linear equations $(A - I_2)(x, y)^T = 0$ has a non-trivial solution. In particular, we see that $v_1 := (2, 3)^T$ is an eigenvector of A corresponding to the eigenvalue 1.

Let $(x, y)^T$ be an eigenvalue of A with eigenvalue 0.75. Then

$$(A - 0.75I_2)(x, y)^T = 0$$

This system of linear equations has augmented matrix

$$\left(A - (0.75)I_3 \mid 0 \right) = \left(\begin{array}{cc|c} 0.1 & 0.1 & 0 \\ 0.15 & 0.15 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

It follows that the system of linear equations $(A - 0.75I_2)(x, y)^T = 0$ has a non-trivial solution. In particular, we see that $v_2 := (-1, 1)^T$ is an eigenvector of A corresponding to the eigenvalue 0.75.

(ii) (Since v_1, v_2 are eigenvectors of A corresponding to distinct eigenvalues of A, they are linearly independent, and so, as there are 2 of them, they form a basis for $\mathbb{R}^2$.) We find $\mu_1, \mu_2 \in \mathbb{R}$ such that $\mu_1 v_1 + \mu_2 v_2 = (0.3, 0.7)^T$ by solving the system of linear equations given, in matrix form, by

$$\left(\begin{array}{c|c} v_1 & v_2 \end{array} \right) \left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right) = \left(\begin{array}{c} 0.3 \\ 0.7 \end{array} \right).$$

The augmented matrix of this system is

$$\left(\begin{array}{cc|c} 2 & 1 & 0.3 \\ 3 & -1 & 0.7 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0.5 & 0.15 \\ 3 & -1 & 0.7 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0.5 & 0.15 \\ 0 & -2.5 & 0.25 \end{array} \right)$$
$$\sim \left(\begin{array}{cc|c} 1 & 0.5 & 0.15 \\ 0 & 1 & -0.1 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0.2 \\ 0 & 1 & -0.1 \end{array} \right).$$

Hence $(1 \ \ 0 \ \ 0)^T = (0.2)v_1 - (0.1)v_2$.

(iii) For each non-negative integer n, let $w_n = (w_{n1} \ \ w_{n2})^T \in \mathbb{R}^2$, where $100w_{n1}$ is the percentage of customers on the route flying with Sheffield air after n years, and $100w_{n2}$ is the percentage with the competition.

The information in the question tells us that $w_0 = (0.3, 0.7)^T$, and $w_{n+1} = Aw_n$ for all $n \in \mathbb{N}_0$.

It therefore follows from the standard theory of difference equations that, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} w_n &= A^n w_0 = A^n \left((0.2)v_1 - (0.1)v_2\right) = (0.2)A^n v_1 - (0.1)A^n v_2 \\ &= (0.2)1^n v_1 - (0.1)(0.75)^n v_2. \end{aligned}$$

To determine the expected situation after 10 years, we take $n = 10$. The expected percentage of the customers with in Sheffield air is

$$\begin{aligned} 100 w_{10,1} &= 100 \left(0.2 \times 1^1 0 \times 2 - 0.1 \times (0.75)^{10} \times 1\right) \\ &\approx 39.4 \end{aligned}$$

approximately.

SOM201 2008 Solution 3

(i)(a) The subset L_1 of $\mathbb{R}^4$ is a subspace of $\mathbb{R}^4$ because it is the null space of the 1×4 row matrix $A := \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$. Since the only row of B is non-zero, it is clear that the rank of A is 1. Therefore $\dim L_1 = \text{nullity}(A) = 4 - \text{rank}(A) = 4 - 1 = 3$.

(b) The subset L_2 does not contain the zero vector $0 = (0, 0, 0, 0)$, and is therefore not a subspace of $\mathbb{R}^4$.

(c) Since the square of a real number is never negative,

$$L_3 = \{(w, x, y, z) \in \mathbb{R}^4 : w = x = y = z = 0\} = \{0\}$$

The dimension of L_3 is zero.

(d) The subset L_4 does not contain the zero vector $0 = (0, 0, 0, 0)$, and is therefore not a subspace of $\mathbb{R}^4$.

(e) Observe $(1, -1, 0, 0) \in L_5$, and $(1, 0, -1, 0) \in L_5$. However

$$(1, -1, 0, 0) + (1, 0, -1, 0) = (2, -1, -1, 0) \notin L_5.$$

So the set L_5 is not a subspace of $\mathbb{R}^4$.

(ii)(a) Note that $\dim W_1 = \text{nullity}(A + I_3)$. We find the nullity of $A + I_3$ by finding the rank of $A + I_3$ and using the Rank-Nullity Theorem. Now

$$A + I_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

so clearly $\text{rank}(A + I_3) = 2$. Hence $\dim W_1 = 3 - 2 = 1$.

(b) Note that $\dim W_2 = \text{nullity}(A - I_4)$. We find the nullity of $A - I_4$ by finding the rank of $A - I_4$ and using the Rank-Nullity Theorem. Now

$$A - I_3 = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

Looking at columns, we see $\text{rank}(A - I_3) = 2$. Hence $\dim W_2 = 3 - 2 = 1$.

(c) Note that $\dim W_3 = \text{nullity}(A)$. We find the nullity of A by finding the rank of A and using the Rank-Nullity Theorem. Now

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence $\text{rank}(A) = 3$, and $\dim W_3 = 3 - 3 = 0$.

Alternatively, we could check that 0 is not an eigenvalue of A.

(d) Now $W_1 \cap W_2 = \{v \in \mathbb{R}^4 : Av = -v \text{ and } Av = v\}$. Therefore $W_1 \cap W_2 = \{(0, 0, 0, 0)\} = \{0_{\mathbb{R}^4}\}$ is the zero subspace of $\mathbb{R}^4$, and so $\dim W_4 = 0$.

(e) The dimension of W_5 is the rank of A. By the calculation in part (c), we have $\dim W_5 = 3$.

SOM201 2008 Solution 4

(i)(a) Observe

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So the rank of A is less than 3. Hence $\det(A) = 0$.

(b) The matrix B is upper triangular. So the determinant is the product of the diagonal entries, and $\det(B) = 6$.

(c) Observe

$$C = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

Further, each of the above EROs take the form of adding a multiple of one row of the matrix to another row, which does not affect the determinant. The final matrix is lower triangular. Hence the determinant of C is the product of the diagonal entries, and $\det(C) = 4$.

(d) We have $\det(D) = \det(B)\det(C) = 6 \times 4 = 24$.

(ii)(a) We saw in part (i) that

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of the last matrix (which is equal to the column rank) is 2. Therefore the matrix A has rank 2.

(b) Since $\det(B) \neq 0$, the matrix B is invertible, and B has rank 4.

(c) As in (b), the matrix C has rank 4.

(d) As in (b), the matrix C has rank 4.

(iii) For $i = 1, 2, 3$, let v_i denote the ith column of P. Set

$$A := \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} \qquad \text{and} \qquad D := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Since $P^{-1}AP = D$, we have $AP = PD$, that is,

$$A \left(\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array} \right) = \left(\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \left(\begin{array}{c|c|c} 1v_1 & 2v_2 & 3v_3 \end{array} \right).$$

Thus $Av_1 = 1v_1$, $Av_2 = 2v_2$ and $Av_3 = 3v_3$, so that 1, 2 and 3 are eigenvalues of A with corresponding eigenvectors v_1, v_2 and v_3 respectively.

Let

$$Y = Y(x) := \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T = \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) \end{pmatrix}^T$$

be the 3×1 column matrix whose entries are the functions y_1, y_2, y_3. The general solution of the given system of linear differential equations is

$$Y = c_1 e^x v_1 + c_2 e^{2x} v_2 + c_3 e^{3x} v_3, \text{ where } c_1, c_2, c_3 \text{ are arbitrary scalars.}$$

We require the solution for which $Y(0) = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}^T$, and so we solve the system of linear equations

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & c_3 & = & 1 \\ -\ c_1 & - & c_2 & - & c_3 & = & 1 \\ & & 2c_2 & + & 2c_3 & = & 2 \end{array}.$$

The augmented matrix of this system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 0 & 2 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Thus $c_1 = -2$, $c_2 = 2$, $c_3 = -1$, and the required solution is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = -2e^x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 2e^{2x} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - e^{3x} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

SOM201 2008 Solution 5

(i)(a)

$$\begin{aligned} Q(x,y,z) &= x^2+2xy+2y^2-2yz \\ &= x^2+2xy+y^2+y^2-2yz+z^2-z^2 \\ &= (x+y)^2+(y-z)^2-z^2 \end{aligned}$$

The matrix

$$P := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

is invertible (since it has non-zero determinant), and so

$$\begin{array}{ccccccc} x' & = & x & + & y & & \\ y' & = & & & y & - & z \\ z' & = & & & & & z \end{array}$$

are three linearly independent linear forms. Therefore

$$Q(x,y,z) = x'^2 + y'^2 - z'^2$$

where x', y', z' are linearly independent linear forms.

(b) The rank of $Q(x,y,z)$ is therefore $2+1=3$ and the signature of $Q(x,y,z)$ is $2-1=1$.

(c) Since

$$\begin{aligned} Q(x,y,z) &= x^2+2xy+2y^2-2yz = (x,y,z)A(x,y,z)^T \\ &= (x,y,z)P^T(P^{-1})^T AP^{-1}P(x,y,z)^T \\ &= (x',y',z')(P^{-1})^T AP^{-1}(x',y',z')^T = (x',y',z')\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \end{aligned}$$

and $(P^{-1})^T AP^{-1}$ is symmetric, we see that $S := P^{-1}$ is a 3×3 matrix (that is invertible because it is the inverse of an invertible matrix) such that $S^T AS$ is a diagonal matrix with entries 1, 1 and -1 along its diagonal. We find P^{-1}:

$$\big(P|I_3\big) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

Therefore

$$S = P^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is an invertible matrix such that

$$S^T AS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D.$$

(c) Notice that $Q(x,y,z) = (x,y,z)A(x,y,z)^T$, where A is as in part (c) of the question. As A is a real symmetric matrix, its eigenvalues are all real, and the conclusions of part (b) mean that A

must have two positive eigenvalues and one negative eigenvalue. Therefore the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x, y, z) = 1$ is a hyperboloid with one sheet.

(ii) We have

$$R(x, y, z) = (x, y, z)B(x, y, z)^T$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{pmatrix}$$

We find the eigenvalues of B. They are the roots of the characteristic polynomial $\chi_B(t)$ of B. Now

$$\chi_B(t) = \begin{vmatrix} 1-t & 0 & 0 \\ 0 & 1-t & -2 \\ 0 & -2 & 1-t \end{vmatrix} = (1-t)\left(1 - 2t + t^2 - 4\right)$$
$$= (1-t)(t^2 - 2t - 3) = (1-t)(t-3)(t+1).$$

Therefore the eigenvalues of B are -1, 3, and 1.

The maximum and minimum values of K are the maximum and the minimum of the eigenvalues of B, namely 3 and -1, respectively.