MAS201 2009 Solution 1

(i) We have

$$A \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So

$$A \sim E =\sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(ii) The rank of A (which is equal to the column rank of E) is 2.

(iii) Since the rank of A is less than 4 (the size of the matrix A), the determinant of A is 0.

(iv) The general solution is

$$x_4 = \lambda,\ x_3 = \mu,\ x_2 = -3\lambda - 2\mu,\ x_1 = 2\lambda + \mu$$

where λ, μ, ν are arbitrary real numbers. Thus, in (column) vector form, the set of all solutions of $AX = 0$ is

$$\left\{ \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\}$$

(v) Let

$$e = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \qquad f = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

The set $\mathcal{N}_A$ is the set of all column vectors $v \in \mathbb{R}^4$ which are solutions of $AX = 0$. By (iv),

$$\mathcal{N}_A = \{\lambda e + \mu f \mid \mu, \nu \in \mathbb{R}\}.$$

Thus $\mathcal{N}_A = \mathrm{Sp}\{e, f\}$ is the span of the vectors e, f, and so is a subspace of $\mathbb{R}^4$. We show that e, f are linearly independent.

Suppose $\alpha, \beta \in \mathbb{R}$ are such that $\alpha e + \beta f = 0$. Looking at row 3, we see $\beta = 0$. Looking at row 4, we see $\alpha = 0$. So e and f are linearly independent.

(vi) Write

$$E = \left(\begin{array}{c|c|c|c} D_1 & D_2 & D_3 & D_4 \end{array} \right),$$

so that D_j is the jth column of E (for $j = 1, \ldots, 4$). Note that

$$A = \left(\begin{array}{c|c|c|c} v_1 & v_2 & v_3 & v_4 \end{array} \right).$$

(a) The dimension of W is the rank of A, which we previously calculated to be 2.

(b) The vectors D_1 and D_2 are linearly independent. So the vectors v_1, v_2 are also linearly independent.

Since the space W has dimension 2, a linearly independent set of 2 vectors is a basis. Thus $\{v_1, v_2\}$ is a basis for W.

MAS201 2008 Solution 2

(i) We have characteristic polynomial

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 1/2 - t & 1/4 & 0 \\ 1/2 & 1/2 - t & 1/2 \\ 0 & 1/4 & 1/2 - t \end{vmatrix}$$

We find

$$\chi_A(t) = (\frac{1}{2} - t)t(t-1).$$

So the matrix A has eigenvalues 0, 1, and $\frac{1}{2}$.

Let X be an eigenvector of A with eigenvalue 0. Then $AX = 0$.

This system of linear equations has augmented matrix

$$\left(\begin{array}{ccc|c} 1/2 & 1/4 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 0 \\ 0 & 1/4 & 1/2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that the system of linear equations $AX = 0$ has a non-trivial solution. In particular, we see that $v_1 := (1, -2, 1)^T$ is an eigenvector of A corresponding to the eigenvalue 0.

Let X be an eigenvector of A with eigenvalue 1. Then $(A - I)X = 0$.

This system of linear equations has augmented matrix

$$\left(\begin{array}{ccc|c} -1/2 & 1/4 & 0 & 0 \\ 1/2 & -1/2 & 1/2 & 0 \\ 0 & 1/4 & -1/2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that the system of linear equations $(A - I)X = 0$ has a non-trivial solution. In particular, we see that $v_2 := (1, 2, 1)^T$ is an eigenvector of A corresponding to the eigenvalue 1.

Let X be an eigenvector of A with eigenvalue $\frac{1}{2}$. Then $(A - \frac{1}{2}I)X = 0$.

This system of linear equations has augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/4 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that the system of linear equations $(A - \frac{1}{2}I)X = 0$ has a non-trivial solution. In particular, we see that $v_3 := (-1, 0, 1)^T$ is an eigenvector of A corresponding to the eigenvalue $\frac{1}{2}$.

(ii) (Since v_1, v_2, v_3 are eigenvectors of A corresponding to distinct eigenvalues of A, they are linearly independent, and so, as there are 3 of them, they form a basis for $\mathbb{R}^3$.) We find $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that $\mu_1 v_1 + \mu_2 v_2 + \mu_3 v_3 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})^T$ by solving the system of linear equations given, in matrix form, by

$$\left(\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array} \right) \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \end{pmatrix}.$$

The augmented matrix of this system is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1/2 \\ -2 & 2 & 0 & 1/4 \\ 1 & 1 & 1 & 1/4 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1/2 \\ 0 & 4 & -2 & 5/4 \\ 0 & 0 & 2 & -1/4 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1/2 \\ 0 & 1 & -1/2 & 5/16 \\ 0 & 0 & 1 & -1/8 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3/8 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/8 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/8 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/8 \end{array} \right).$$

Hence $(1/2, 1/4/1/4)^T = \frac{1}{8}v_1 + \frac{1}{4}v_2 - \frac{1}{8}v_3$.

(iii) For each non-negative integer n, let $w_n = (w_{n1}, w_{n2}, w_{n3})^T \in \mathbb{R}^2$, where w_{n1} is the proportion of cars in Aberdeen after n months, w_{n2} the proportion in Birmingham, and w_{n3} the proportion in Chesterfield.

The information in the question tells us that $w_0 = (1/2, 1/4, 1/4)^T$, and $w_{n+1} = Aw_n$ for all $n \in \mathbb{N}_0$.

It therefore follows from the standard theory of difference equations that, for all $n \in \mathbb{N}_0$,

$$w_n = A^n w_0 = A^n \left(\frac{1}{8}v_1 + \frac{1}{4}v_2 - \frac{1}{8}v_3 \right) = \frac{1}{8}A^n v_1 + \frac{1}{4}A^n v_2 - \frac{1}{8}A^n v_3 = \frac{1}{4}v_2 - \frac{1}{8(2^n)}v_3.$$

To determine the situation after 3 months, we take $n = 3$. We see that the proportions of cars in Aberdeen, Birmingham, and Chesterfield are the entries in the vector

$$\frac{1}{4}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{64}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 17/64 \\ 1/2 \\ 15/64 \end{pmatrix}$$

MAS201 2009 Solution 3

(i)(a) False, (b) True, (c) True

(ii)(a) Observe $(1,0)^T \in L_1$, but $-(1,0) = (-1,0) \notin L_1$. So L_1 is not a subspace of $\mathbb{R}^2$.

(b) The set L_2 is the null space of the matrix $(1 \;\; 1)$. So L_2 is a subspace of $\mathbb{R}^2$.

(c) The set L_3 is the null space of the matrix $(1 \quad -1)$. So L_2 is a subspace of $\mathbb{R}^2$.

(d) Observe $(1,1)^T \in L_4$, and $(1,-1)^T \in L_4$. But $(1,1)+(1,-1) = (2,0) \notin L_4$. So L_4 is not a subspace of $\mathbb{R}^2$.

(e) Observe $L_5 = L_3$. So L_5 is a subspace of $\mathbb{R}^2$.

(iii) By a sequence of elementary row operations, we have

$$A \sim E := \begin{pmatrix} 1 & 0 & -4 & 7 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) The set V is the column space of the matrix A, making it a subspace. Its dimension is the rank of A, which is 2.

(b) The first two columns of the matrix E are linearly independent. Therefore the first two columns of the matrix A are linearly independent. Both of these columns belong to the 2-dimensional vector space V; by linear independence they form a basis.

So we have a basis consisting of the two vectors $(4,6,3)^T$ and $(5,5,4)^T$.

(c) By the rank-nullity theorem, the subspace $\mathcal{N}_A$ has dimension $4 - \text{rank}(A) = 2$.

MAS201 2009 Solution 4

(i) We have characteristic polynomial

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 2-t & 1 & 1 \\ 1 & 3-t & 3 \\ 3 & 2 & 2-t \end{vmatrix}$$

We find

$$\chi_A(t) = -t(t-1)(t-6).$$

So the matrix A has eigenvalues 0, 1, and 6.

Let X be an eigenvector of A with eigenvalue 0. Then $AX = 0$.

This system of linear equations has augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 3 & 3 & 0 \\ 3 & 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & -7 & -7 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that the system of linear equations $AX = 0$ has a non-trivial solution. In particular, we see that $v_1 := (0, -1, 1)^T$ is an eigenvector of A corresponding to the eigenvalue 0.

Let X be an eigenvector of A with eigenvalue 1. Then $(A - I)X = 0$.

This system of linear equations has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that the system of linear equations $(A - I)X = 0$ has a non-trivial solution. In particular, we see that $v_2 := (1, -2, 1)^T$ is an eigenvector of A corresponding to the eigenvalue 1.

Let X be an eigenvector of A with eigenvalue 6. Then $(A - 6I)X = 0$.

This system of linear equations has augmented matrix

$$\left(\begin{array}{ccc|c} -4 & 1 & 1 & 0 \\ 1 & -3 & 3 & 0 \\ 3 & 2 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & -3 & 0 \\ -4 & 1 & 1 & 0 \\ 3 & 2 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & -3 & 0 \\ 0 & -11 & 13 & 0 \\ 0 & 11 & -13 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -3 & -3 & 0 \\ 0 & 1 & 13/11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -6/11 & 0 \\ 0 & 1 & -13/11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that the system of linear equations $(A - \frac{1}{2}I)X = 0$ has a non-trivial solution. In particular, we see that $v_3 := (6, 13, 11)^T$ is an eigenvector of A corresponding to the eigenvalue 6.

(ii) Let

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

and

$$P = \left(\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array} \right) = \left(\begin{array}{c|c|c} 0 & 1 & 6 \\ -1 & -2 & 13 \\ 1 & 1 & 11 \end{array} \right).$$

Then $P^{-1}AP = D$.

(iii) Let

$$Y = Y(x) := \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T = \begin{pmatrix} y_1(x) & y_2(x) & y_3(x) \end{pmatrix}^T.$$

The general solution of the given system of linear differential equations is

$$Y(x) = c_1 v_1 + c_2 e^x v_2 + c_3 e^{6x} v_3, \text{ where } c_1, c_2, c_3 \text{ are arbitrary scalars.}$$

We require the solution for which $Y(0) = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T = v_2$.

Setting $x = 0$, we see $c_1 v_1 + c_2 v_2 + c_3 v_3 = v_2$.

Since the vectors v_1, v_2, v_3 are linearly independent, we find $c_1 = 0$, $c_2 = 1$, and $c_3 = 0$.

So we have solution $Y(0) = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T e^t$.

MAS201 2009 Solution 5

(i)

$$
\begin{aligned}
Q(x,y,z) &= x^2+2y^2+z^2+2xy-2xz-4yz \\
&= (x^2+2xy-2xz)+2y^2-4yz+z^2 \\
&= (x+y-z)^2-(y^2+z^2-2yz)+2y^2-4yz+z^2 \\
&= (x+y-z)^2+y^2-2yz+z^2-z^2 \\
&= (x+y-z)^2+(y-z)^2-z^2
\end{aligned}
$$

The matrix

$$P := \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

is invertible (since it has non-zero determinant), and so

$$
\begin{array}{ccccccc}
x' & = & x & + & y & - & z \\
y' & = & & & y & - & z \\
z' & = & & & & & z
\end{array}
$$

are three linearly independent linear forms, with $Q(x,y,z) = x'^2 + y'^2 - z'^2$.

(ii) The rank of $Q(x,y,z)$ is therefore $2+1=3$ and the signature of $Q(x,y,z)$ is $2-1=1$.

(iii) Notice that $Q(x,y,z) = (x,y,z)A(x,y,z)^T$, where A is as above. As A is a real symmetric matrix, its eigenvalues are all real, and the conclusions of part (ii) mean that A must have two positive eigenvalues and one negative eigenvalue. Therefore the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x,y,z)=1$ is a hyperboloid with one sheet.

(iv) Let $S = P^{-1}$. Then

$$S^TAS = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We find P^{-1}:

$$\left(P|I_3\right) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right.$$

Therefore

$$S = P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(v) We have

$$R(x,y,z) = (x,y,z)B(x,y,z)^T$$

where

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 1 \end{pmatrix}$$

We find the eigenvalues of B. They are the roots of the characteristic polynomial $\chi_B(t)$ of B. Now

$$\chi_B(t) = \begin{vmatrix} 1-t & 1 & 0 \\ 1 & 2-t & -1 \\ 0 & -1 & 1-t \end{vmatrix}$$

We find $\chi_B(t) = t(1-t)(t-3)$.

Therefore the eigenvalues of B are 0, 1, and 3.

The maximum and minimum values of K are the maximum and the minimum of the eigenvalues of B, namely 3 and 0, respectively.