The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Linear Mathematics for Applications **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. ***(3 marks)***

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Row-reduce the following matrix. ***(6 marks)***

$$D = \begin{bmatrix} 11 & 10 & 1 & 1 & 11 \\ 11 & 1 & 10 & 10 & 1 \\ 1 & 1 & 0 & 0 & 10 \end{bmatrix}$$

(c) You may assume the row-reduction

$$\begin{bmatrix} 7 & -3 & 1 & -1 & 1 \\ 3 & 2 & 7 & 16 & 16 \\ 4 & -1 & 2 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve the following two systems of equations (the first system on the left, and the second system on the right) :

$$\begin{aligned} 7x - 3y + z &= -1 \\ 3x + 2y + 7z &= 16 \\ 4x - y + 2z &= 3 \end{aligned} \qquad\qquad \begin{aligned} 7x - 3y + z &= 1 \\ 3x + 2y + 7z &= 16 \\ 4x - y + 2z &= -3 \end{aligned}$$

In each case say whether the system has a unique solution, an infinite family of solutions, or no solution. ***(6 marks)***

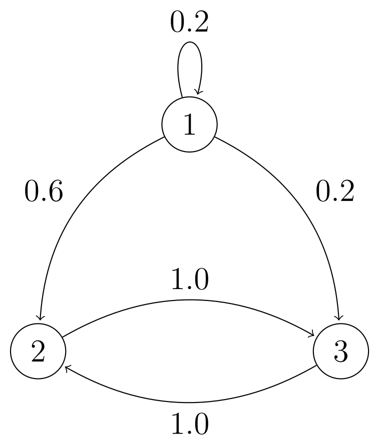
(d) Find the determinant of the following matrix: ***(3 marks)***

$$E = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 4 & 0 & 5 \end{bmatrix}$$

(e) State, with justification, which of the following matrices are invertible. ***(7 marks)***

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 9 & 9 & 9 & 9 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 2 & 5 \\ 6 & 4 & 3 \\ 5 & 1 & 2 \\ 7 & 9 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} -2 & -2 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2 Consider the following Markov chain:



(a) Write down the associated transition matrix. ***(2 marks)***

(b) Find a stationary distribution for the system. ***(6 marks)***

(c) If the system is in state 1 at $t = 0$, what is the probability that it is in state 2 at $t = 4$? ***(17 marks)***

3 (1) Are the following statements true or false? Justify your answers carefully. *(9 marks)*

(a) Any list of four vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$.

(b) There exists a linearly dependent list of vectors that spans $\mathbb{R}^3$.

(c) The following vectors are linearly independent:

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \quad \begin{bmatrix}2\\2\\3\\4\end{bmatrix} \quad \begin{bmatrix}3\\3\\3\\4\end{bmatrix} \quad \begin{bmatrix}4\\4\\4\\4\end{bmatrix} \quad \begin{bmatrix}4\\3\\2\\1\end{bmatrix}$$

(d) The following vectors form a basis of $\mathbb{R}^3$:

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}1\\10\\100\end{bmatrix} \quad \begin{bmatrix}1\\11\\101\end{bmatrix}$$

(2) Which of the following sets is a subspace of $\mathbb{R}^4$? Justify your answers. *(9 marks)*

$$\begin{aligned}
V_1 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w+x+y+z=0\} \\
V_2 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w^2+x^2+y^2+z^2=0\} \\
V_3 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w^3+x^3+y^3+z^3=0\} \\
V_4 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w+x+y+z=1\}.
\end{aligned}$$

(3) Give examples of the following. *(7 marks)*

(a) A list of 4 vectors in $\mathbb{R}^3$ such that any three of them form a basis.

(b) A pair of subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 3$ and $\dim(V+W) = 4$.

(c) A list of three subspaces $P, Q, R \leq \mathbb{R}^3$ such that $\dim(P) = \dim(Q) = \dim(R) = 2$ and $\dim(P \cap Q \cap R) = 1$.

4 Put

$$v_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \quad v_2 = \begin{bmatrix}2\\2\\-1\\0\end{bmatrix} \quad u_1 = \begin{bmatrix}1\\2\\2\\-1\end{bmatrix} \quad u_2 = \begin{bmatrix}2\\2\\2\\-1\end{bmatrix}$$

and $V = \operatorname{span}(v_1, v_2)$ and $W = \operatorname{ann}(u_1, u_2)$.

(a) Find the canonical basis for V. *(4 marks)*

(b) Find the canonical basis for W. *(6 marks)*

(c) Find the canonical basis for $V + W$. *(5 marks)*

(d) Find vectors c_1 and c_2 such that $V = \operatorname{ann}(c_1, c_2)$. *(5 marks)*

(e) Find the canonical basis for $V \cap W$. *(5 marks)*

5 Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}.$$

You may assume that $\det(A - tI) = t^4 - 4t^3 - 12t^2$.

(a) Find the eigenvalues of A. ***(2 marks)***

(b) Find an orthonormal basis of $\mathbb{R}^4$ consisting of eigenvectors of A. ***(14 marks)***

(c) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. ***(4 marks)***

(d) Express the quadratic form

$$Q = w^2 + x^2 + y^2 + z^2 + 2(wx + yz) + 4(wy + wz + xy + xz)$$

as $Q = F^2 - G^2$, where F and G are linear forms. Hence express Q as a product of two linear forms. ***(5 marks)***

End of Question Paper