Linear Mathematics for Applications — Exam

(1)

(a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. **(3 marks)**

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Row-reduce the following matrix. **(6 marks)**

$$D = \begin{bmatrix} 11 & 10 & 1 & 1 & 11 \\ 11 & 1 & 10 & 10 & 1 \\ 1 & 1 & 0 & 0 & 10 \end{bmatrix}$$

(c) You may assume the row-reduction

$$\begin{bmatrix} 7 & -3 & 1 & -1 & 1 \\ 3 & 2 & 7 & 16 & 16 \\ 4 & -1 & 2 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve the following two systems of equations (the first system on the left, and the second system on the right) :

$$\begin{aligned} 7x - 3y + z &= -1 \\ 3x + 2y + 7z &= 16 \\ 4x - y + 2z &= 3 \end{aligned} \qquad\qquad \begin{aligned} 7x - 3y + z &= 1 \\ 3x + 2y + 7z &= 16 \\ 4x - y + 2z &= -3 \end{aligned}$$

In each case say whether the system has a unique solution, an infinite family of solutions, or no solution. **(6 marks)**

(d) Find the determinant of the following matrix: **(3 marks)**

$$E = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 4 & 0 & 5 \end{bmatrix}$$

(e) State, with justification, which of the following matrices are invertible. **(7 marks)**

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 9 & 9 & 9 & 9 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 2 & 5 \\ 6 & 4 & 3 \\ 5 & 1 & 2 \\ 7 & 9 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} -2 & -2 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution:

(a) None of the matrices are in RREF. The matrix A is not in RREF because it has a row of zeros that does not occur after all the nonzero rows.[1]The matrix B is not in RREF because the pivot in the second row is to the left of the pivot in the first row.[1]The matrix C is not in RREF because there are nonzero entries above the pivot in the third row.[1]

(b)

$$\begin{bmatrix} 11 & 10 & 1 & 1 & 11 \\ 11 & 1 & 10 & 10 & 1 \\ 1 & 1 & 0 & 0 & 10 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 1 & 1 & -99 \\ 0 & -10 & 10 & 10 & -109 \\ 1 & 1 & 0 & 0 & 10 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -1 & -1 & 99 \\ 0 & 0 & 0 & 0 & 881 \\ 1 & 0 & 1 & 1 & -89 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. [6]$$

(c) The left hand system corresponds to the first augmented matrix shown below:

$$\left[\begin{array}{ccc|c} 7 & -3 & 1 & -1 \\ 3 & 2 & 7 & 16 \\ 4 & -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By deleting the last column from the row-reduction given in the question, we see that our matrix row-reduces as indicated, so the left hand system is equivalent to the system

$$x + z = 2 \qquad y + 2z = 5 \qquad 0 = 0.[1]$$

The solutions have the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-z \\ 5-2z \\ z \end{bmatrix}$ with z arbitrary [1]. In particular, there are infinitely many solutions, one for each possible value of z [1].

Similarly, we can delete the fourth column from the given row-reduction to get

$$\left[\begin{array}{ccc|c} 7 & -3 & 1 & 1 \\ 3 & 2 & 7 & 16 \\ 4 & -1 & 2 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This shows that the right-hand system is equivalent to the system

$$x + z = 0 \qquad y + 2z = 0 \qquad 0 = 1, [2]$$

so there are no solutions [1].

(d) There are enough zeros in E that a direct expansion is painless:

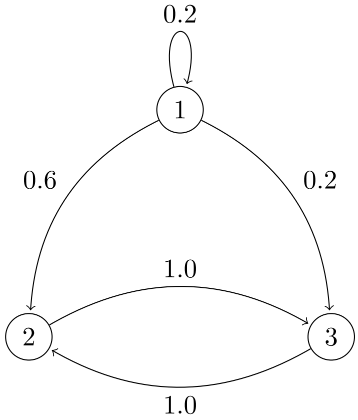
$$\begin{aligned} \det(E) &= \det \begin{bmatrix} 0 & 3 & 0 \\ 4 & 0 & 4 \\ 4 & 0 & 5 \end{bmatrix} + 2\det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 5 \end{bmatrix} [1] \\ &= -3\det \begin{bmatrix} 4 & 4 \\ 4 & 5 \end{bmatrix} + 2 \times 0 [1] \\ &= -3 \times (20-16) = -12. [1] \end{aligned}$$

(e) In matrix F the last row is 9 times the first row, so the rows are linearly dependent, so F is not invertible [2]. The matrix G is not square and so cannot be invertible [1]. Next, the matrix H can be row-reduced to the identity as follows:

$$\begin{bmatrix} -2 & -2 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & -2 & -3 \\ 1 & 0 & -1 \\ 0 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & -2 & -3 \\ 1 & 0 & -1 \\ 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This shows that H is invertible. Alternatively, we can calculate that $\det(H) = -12 \neq 0$, which also implies that H is invertible [2]. Finally, the matrix is J is also invertible. One of many ways to see this is to note that $J^2 = I$, so J is its own inverse. [2]

(2) Consider the following Markov chain:



(a) Write down the associated transition matrix. **(2 marks)**

(b) Find a stationary distribution for the system. **(6 marks)**

(c) If the system is in state 1 at $t = 0$, what is the probability that it is in state 2 at $t = 4$? **(17 marks)**

Solution:

(a) $P = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.6 & 0 & 1 \\ 0.2 & 1 & 0 \end{bmatrix}$. [2]

(b) A stationary distribution p is in particular an eigenvector of eigenvalue 1. Such eigenvectors can be found by row-reducing $P - I$ [1]:

$$\begin{bmatrix} -0.8 & 0 & 0 \\ 0.6 & -1 & 1 \\ 0.2 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ [2]}$$

We conclude that if $p = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^T$ then we must have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or equivalently $p_1 = 0$ and $p_2 = p_3$ [1]. For a stationary distribution we also need $p_1, p_2, p_3 \geq 0$ and $p_1 + p_2 + p_3 = 1$, so we must take $p_2 = p_3 = 0.5$ [1]. It follows that the stationary distribution is $\begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}^T$ [1].

(c) We are given that the initial distribution is $r_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$, and we need to calculate $r_4 = P^4 r_0$.

As 4 is not very large, it is easy enough to do this directly:

$$P^2 = \begin{bmatrix} -0.8 & 0 & 0 \\ 0.6 & -1 & 1 \\ 0.2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -0.8 & 0 & 0 \\ 0.6 & -1 & 1 \\ 0.2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0.04 & 0 & 0 \\ 0.32 & 1 & 0 \\ 0.64 & 0 & 1 \end{bmatrix}$$

$$P^4 = (P^2)^2 = \begin{bmatrix} 0.04 & 0 & 0 \\ 0.32 & 1 & 0 \\ 0.64 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.04 & 0 & 0 \\ 0.32 & 1 & 0 \\ 0.64 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.0016 & 0 & 0 \\ 0.3328 & 1 & 0 \\ 0.6656 & 0 & 1 \end{bmatrix}$$

$$r_4 = P^4 r_0 = \begin{bmatrix} 0.0016 & 0 & 0 \\ 0.3328 & 1 & 0 \\ 0.6656 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0016 \\ 0.3328 \\ 0.6656 \end{bmatrix}$$

The probability of being in state 2 at $t = 4$ is the second component of r_4, which is 0.3328. Full credit will be given for this approach.

However, most students will probably use the following method. We first find the remaining eigenvalues and eigenvectors for P [1]. The characteristic polynomial is

$$\chi_P(t) = \det \begin{bmatrix} 0.2-t & 0 & 0 \\ 0.6 & -t & 1 \\ 0.2 & 1 & -t \end{bmatrix} = (0.2-t)(t^2-1)\text{[1]} = (t-1)(t+1)(0.2-t),$$

so the eigenvalues are 1 and -1 and 0.2 [1]. We have already found an eigenvector $u_1 = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}^T$ of eigenvalue 1. To find an eigenvector of eigenvalue -1, we row-reduce $P+I$:

$$\begin{bmatrix} 1.2 & 0 & 0 \\ 0.6 & 1 & 1 \\ 0.2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{[1]}$$

This shows that the eigenvectors of eigenvalue -1 are the vectors of the form $\begin{bmatrix} x & y & z \end{bmatrix}$ with $x = y + z = 0$, or in other words the vectors of the form $\begin{bmatrix} 0 & y & -y \end{bmatrix}$ [1]. For compatibility with u_1 it will be convenient to take $y = 0.5$ giving $u_2 = \begin{bmatrix} 0 & 0.5 & -0.5 \end{bmatrix}$ [1]. Next, to find an eigenvector of eigenvalue 0.2 we row-reduce $P - 0.2I$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0.6 & -0.2 & 1 \\ 0.2 & 1 & -0.2 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 \\ 0.6 & -0.2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 \\ 0 & -3.2 & 1.6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \text{[2]}$$

From this we see that a suitable eigenvector is $u_3 = \begin{bmatrix} -1.5 & 0.5 & 1 \end{bmatrix}$ [1]. We next need to write the vector $r_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ as a linear combination of the eigenvectors u_i. We can do this by row-reducing the matrix $[u_1|u_2|u_3|r_0]$: [1]

$$\begin{bmatrix} 0 & 0 & -1.5 & 1 \\ 0.5 & 0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1.5 & 1 \\ 0.5 & 0.5 & 0.5 & 0 \\ 0 & -1 & 0.5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 2/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -2/3 \end{bmatrix} \text{[2]}$$

The required coefficients appear in the last column, so $r_0 = u_1 - \frac{1}{3}u_2 - \frac{2}{3}u_3$ [1]. This gives

$$r_4 = P^4 r_0 = P^4 u_1 - \tfrac{1}{3}P^4 u_2 - \tfrac{2}{3}P^4 u_3\text{[1]} = 1^4 u_1 - \tfrac{1}{3}(-1)^4 u_2 - \tfrac{2}{3}(0.2)^4 u_3\text{[1]}$$

$$= u_1 - \tfrac{1}{3}u_2 - \tfrac{2}{3}(0.2)^4 u_3 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/6 \\ 1/6 \end{bmatrix} + (0.2)^4 \begin{bmatrix} 1 \\ -1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 0.0016 \\ 0.3328 \\ 0.6656 \end{bmatrix} \text{[1]}$$

Again, the probability of being in state 2 at $t = 4$ is the second component of r_4, which is 0.3328. [1]

(3)

(1) Are the following statements true or false? Justify your answers carefully. **(9 marks)**

(a) Any list of four vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$.

(b) There exists a linearly dependent list of vectors that spans $\mathbb{R}^3$.

(c) The following vectors are linearly independent:

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \qquad \begin{bmatrix}2\\2\\3\\4\end{bmatrix} \qquad \begin{bmatrix}3\\3\\3\\4\end{bmatrix} \qquad \begin{bmatrix}4\\4\\4\\4\end{bmatrix} \qquad \begin{bmatrix}4\\3\\2\\1\end{bmatrix}$$

(d) The following vectors form a basis of $\mathbb{R}^3$:

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \begin{bmatrix}1\\10\\100\end{bmatrix} \qquad \begin{bmatrix}1\\11\\101\end{bmatrix}$$

(2) Which of the following sets is a subspace of $\mathbb{R}^4$? Justify your answers. **(9 marks)**

$$\begin{aligned}
V_1 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w + x + y + z = 0\} \\
V_2 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 0\} \\
V_3 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w^3 + x^3 + y^3 + z^3 = 0\} \\
V_4 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w + x + y + z = 1\}.
\end{aligned}$$

(3) Give examples of the following. **(7 marks)**

(a) A list of 4 vectors in $\mathbb{R}^3$ such that any three of them form a basis.

(b) A pair of subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 3$ and $\dim(V + W) = 4$.

(c) A list of three subspaces $P, Q, R \leq \mathbb{R}^3$ such that $\dim(P) = \dim(Q) = \dim(R) = 2$ and $\dim(P \cap Q \cap R) = 1$.

Solution:

(1) (a) This is false [1]. For the most extreme example, the list $0, 0, 0, 0$ clearly does not span $\mathbb{R}^3$. For a less degenerate example in which the four vectors are all different, we can take $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}, \begin{bmatrix}4\\0\\0\end{bmatrix}$ [1].

(b) This is true [1]. The simplest example is the list e_1, e_2, e_3, 0 [1].

(c) This is false [1]: any list of five vectors in $\mathbb{R}^4$ is automatically linearly dependent [1]. If we call the vectors v_1 to v_5, then we have

$$4v_1 - 5v_4 + 4v_5 = \begin{bmatrix}4\\8\\12\\16\end{bmatrix} - \begin{bmatrix}20\\20\\20\\20\end{bmatrix} + \begin{bmatrix}16\\12\\8\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

which is a specific example of a nontrivial linear relation.

(d) This is true [1]. One way to prove it is to take the vectors as the columns of a matrix, and start row-reducing it:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 10 & 11 \\ 1 & 100 & 101 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 9 & 10 \\ 0 & 99 & 100 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 9 & 10 \\ 0 & 0 & -10 \end{bmatrix}$$

We now have an upper triangular matrix with nonzero entries on the diagonal, and any such matrix can be row-reduced to the identity. It follows that the given list is a basis [2].

(2) (a) The set V_1 is a subspace [1]. Indeed, it is clear that the zero vector lies in V_1. Next, suppose we have two elements $a = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ and $a' = \begin{bmatrix} w' & x' & y' & z' \end{bmatrix}^T$ in V_1, so $w + x + y + z = 0$ and $w' + x' + y' + z' = 0$. By adding these equations, we see that $(w + w') + (x + x') + (y + y') + (z + z') = 0$, so the vector $a + a' = \begin{bmatrix} w + w' & x + x' & y + y' & z + z' \end{bmatrix}$ also lies in V_1. This shows that V_1 is closed under addition. Similarly, for any $t \in \mathbb{R}$ we have $tw + tx + ty + tz = 0$, showing that $ta \in V_1$. This means that V_1 is closed under scalar multiplication and so is a subspace [2].

(b) The set V_2 is a subspace [1]. Indeed, as all squares are nonnegative, the only way we can have $w^2 + x^2 + y^2 + z^2 = 0$ is if $w = x = y = z = 0$. Thus $V_2 = \{0\}$, which is clearly a subspace [1].

(c) The set V_3 is not a subspace [1]. Indeed, the vectors $a = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$ and $a' = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$ lie in V_3, but the sum $a + a' = \begin{bmatrix} 2 & -1 & -1 & 0 \end{bmatrix}^T$ does not (because $2^3 + (-1)^3 + (-1)^3 + 0^3 = 6 \neq 0$) so V_3 is not closed under addition [1].

(d) The set V_4 is not a subspace [1], because it does not contain the zero vector [1].

(3) (a) The simplest example is the list $e_1, e_2, e_3, e_1 + e_2 + e_3$. [2]

(b) The simplest example is to take $V = \text{span}(e_1, e_2, e_3)$ and $W = \text{span}(e_1, e_2, e_4)$ so $V + W = \text{span}(e_1, e_2, e_3, e_4)$. [2]

(c) We need three planes that meet along a common line. One example is to take

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y = z \right\} \qquad Q = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y = 0 \right\} \qquad R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y = -z \right\},$$

so

$$P \cap Q \cap R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y = z = 0 \right\} = \text{ the } x\text{-axis }.$$ [3]

(4) Put

$$v_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}2\\2\\-1\\0\end{bmatrix} \qquad u_1 = \begin{bmatrix}1\\2\\2\\-1\end{bmatrix} \qquad u_2 = \begin{bmatrix}2\\2\\2\\-1\end{bmatrix}$$

and $V = \text{span}(v_1, v_2)$ and $W = \text{ann}(u_1, u_2)$.

(a) Find the canonical basis for V. **(4 marks)**

(b) Find the canonical basis for W. **(6 marks)**

(c) Find the canonical basis for $V + W$. **(5 marks)**

(d) Find vectors c_1 and c_2 such that $V = \text{ann}(c_1, c_2)$. **(5 marks)**

(e) Find the canonical basis for $V \cap W$. **(5 marks)**

Solution:

(a) To find the canonical basis for V we perform the following row-reduction:

$$\begin{bmatrix} v_1^T \\ \hline v_2^T \end{bmatrix} [1] = \begin{bmatrix}1&1&0&1\\2&2&-1&0\end{bmatrix} \to \begin{bmatrix}1&1&0&1\\0&0&-1&-2\end{bmatrix} \to \begin{bmatrix}1&1&0&1\\0&0&1&2\end{bmatrix} [1]$$

From this we see that the canonical basis is (a_1, a_2) where a_1 and a_2 are the transposes of the rows of the above matrix, namely

$$a_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \qquad a_2 = \begin{bmatrix}0\\0\\1\\2\end{bmatrix}. [2]$$

(b) Next, W is the set of vectors x satisfying $x.u_1 = 0$ and $x.u_2 = 0$, or equivalently

$$\begin{aligned} -x_4 + 2x_3 + 2x_2 + 2x_1 &= 0 \\ -x_4 + 2x_3 + 2x_2 + x_1 &= 0 [2] \end{aligned}$$

Solving these in the standard way gives $x_1 = 0$ and $x_4 = 2x_3 + 2x_2$ with x_3 and x_2 arbitrary [1], so

$$x = \begin{bmatrix}0\\x_2\\x_3\\2x_3+2x_2\end{bmatrix} = x_2 \begin{bmatrix}0\\1\\0\\2\end{bmatrix} + x_3 \begin{bmatrix}0\\0\\1\\2\end{bmatrix}. [1]$$

From this we see that the following matrices form the canonical basis for W:

$$b_1 = \begin{bmatrix}0\\1\\0\\2\end{bmatrix} \qquad b_2 = \begin{bmatrix}0\\0\\1\\2\end{bmatrix}. [2]$$

(c) It follows that $V + W = \text{span}(a_1, a_2, b_1, b_2)$ [1], but we can omit b_2 because it is the same as a_2. To make this canonical we perform the following row-reduction:

$$\begin{bmatrix} a_1^T \\ \hline a_2^T \\ \hline b_1^T \end{bmatrix} [2] = \begin{bmatrix}1&1&0&1\\0&0&1&2\\0&1&0&2\end{bmatrix} \to \begin{bmatrix}1&0&0&-1\\0&0&1&2\\0&1&0&2\end{bmatrix} \to \begin{bmatrix}1&0&0&-1\\0&1&0&2\\0&0&1&2\end{bmatrix}. [1]$$

This shows that the canonical basis for $V + W$ consists of the vectors

$$\begin{bmatrix}1&0&0&-1\end{bmatrix}^T \qquad \begin{bmatrix}0&1&0&2\end{bmatrix}^T \qquad \begin{bmatrix}0&0&1&2\end{bmatrix}^T. [1]$$

(d) If x is a vector that annihilates the space $V = \text{span}(v_1, v_2)$, we must have $x.v_1 = x.v_2 = 0$, or equivalently

$$x_4 + x_2 + x_1 = 0$$
$$-x_3 + 2x_2 + 2x_1 = 0.\text{[1]}$$

This gives $x_4 = -x_2 - x_1$ and $x_3 = 2x_2 + 2x_1$ with x_1 and x_2 arbitrary, so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ 2x_1 + 2x_2 \\ -x_1 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}.\text{[2]}$$

The standard methods now tell us that $V = \text{ann}(c_1, c_2)$, where

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \qquad c_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}.\text{[2]}$$

(e) We now have $V \cap W = \text{ann}(c_1, c_2) \cap \text{ann}(u_1, u_2) = \text{ann}(c_1, c_2, u_1, u_2)$ [1]. In other words, $V \cap W$ is the set of solutions to the equations $x.c_1 = 0$ and $x.c_2 = 0$ and $x.u_1 = 0$ and $x.u_2 = 0$, or equivalently

$$-x_4 + 2x_3 + x_1 = 0$$
$$-x_4 + 2x_3 + x_2 = 0$$
$$-x_4 + 2x_3 + 2x_2 + 2x_1 = 0$$
$$-x_4 + 2x_3 + 2x_2 + x_1 = 0.\text{[1]}$$

Subtracting the first two equations gives $x_1 = x_2$, and subtracting the last two gives $x_1 = 0$, so we also have $x_2 = 0$. Given this, everything else reduces easily to the equation $x_4 = 2x_3$ [1]. We thus have

$$x = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 2x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} = x_3 a_2.\text{[1]}$$

It follows that the vector a_2 on its own is the canonical basis for $V \cap W$. [1]

(Note: this could have been obtained more directly. From parts (a) and (b) it is clear that the vector a_2 (which is the same as b_2) lies in $V \cap W$, so the subspace $\mathbb{R}a_2$ is contained in $V \cap W$. If $V \cap W$ were any bigger than this, it would have dimension 2 and so would be the same as V and W, so we would have $V = W$, which is false because the canonical bases of V and W are different. We must therefore have $V \cap W = \mathbb{R}a_2$.)

(5) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}.$$

You may assume that $\det(A - tI) = t^4 - 4t^3 - 12t^2$.

(a) Find the eigenvalues of A. **(2 marks)**

(b) Find an orthonormal basis of $\mathbb{R}^4$ consisting of eigenvectors of A. **(14 marks)**

(c) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. **(4 marks)**

(d) Express the quadratic form

$$Q = w^2 + x^2 + y^2 + z^2 + 2(wx + yz) + 4(wy + wz + xy + xz)$$

as $Q = F^2 - G^2$, where F and G are linear forms. Hence express Q as a product of two linear forms. **(5 marks)**

Solution:

(a) The characteristic polynomial factorises as $\chi_A(t) = t^2(t^2 - 4t - 12) = t^2(t-6)(t+2)$, so the eigenvalues are 0, 6 and -2. [2]

(b) It is easy to see that a vector $\begin{bmatrix} w & x & y & z \end{bmatrix}^T$ is an eigenvector of eigenvalue 0 if and only if $w + x + 2y + 2z = 2w + 2x + y + z = 0$ [2], which reduces to $x = -w$ and $z = -y$ [1]. If we put

$$u_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \end{bmatrix}^T \qquad\qquad u_2 = \begin{bmatrix} 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

then u_1 and u_2 are orthonormal and are eigenvectors of eigenvalue 0 [2].

Next, to find an eigenvector of eigenvalue 6 we row-reduce the matrix $A - 6I$ [1]:

$$\begin{bmatrix} -5 & 1 & 2 & 2 \\ 1 & -5 & 2 & 2 \\ 2 & 2 & -5 & 1 \\ 2 & 2 & 1 & -5 \end{bmatrix} \to \begin{bmatrix} 0 & -24 & 12 & 12 \\ 1 & -5 & 2 & 2 \\ 0 & 12 & -9 & -3 \\ 0 & 12 & -3 & -9 \end{bmatrix} \to \begin{bmatrix} 0 & -24 & 12 & 12 \\ 1 & -5 & 2 & 2 \\ 0 & 12 & -9 & -3 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & -24 & 0 & 24 \\ 1 & -5 & 0 & 4 \\ 0 & 12 & 0 & -12 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} [2].$$

From this we see that the eigenvectors of eigenvalue 6 are the vectors of the form $a = \begin{bmatrix} w & w & w & w \end{bmatrix}^T$ [1]. We note that $\|a\| = \sqrt{w^2 + w^2 + w^2 + w^2} = \sqrt{4w^2} = 2|w|$. To get a unit vector we take $w = 1/2$ giving

$$u_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T . [1]$$

Finally, to find an eigenvector of eigenvalue -2 we row-reduce the matrix $A + 2I$:

$$\begin{bmatrix} 3 & 1 & 2 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 2 & 3 & 1 \\ 2 & 2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & -8 & -4 & -4 \\ 1 & 3 & 2 & 2 \\ 0 & -4 & -1 & -3 \\ 0 & -4 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1/2 & 1/2 \\ 1 & 3 & 2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{[2]}$$

From this we see that the eigenvectors of eigenvalue -2 are the vectors $\begin{bmatrix} w & x & y & z \end{bmatrix}^T$ satisfying $w+z = x+z = y-z = 0$, or in other words the vectors of the form $\begin{bmatrix} w & w & -w & -w \end{bmatrix}^T$ [1]. To get a unit vector we again take $w = 1/2$, giving

$$u_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T . \text{[1]}$$

It is standard that when a and b are eigenvectors of a symmetric matrix with distinct eigenvalues, we have $a.b = 0$. This gives $u_1.u_3 = u_1.u_4 = u_2.u_3 = u_2.u_4 = u_3.u_4 = 0$, and the remaining identity $u_1.u_2 = 0$ is clear by inspection (as are the others, in fact). Thus, we have an orthonormal basis of eigenvectors.

(c) The general theory tells us that we can take

$$P = \left[\begin{array}{c|c|c|c} u_1 & u_2 & u_3 & u_4 \end{array} \right] = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/2 & 1/2 \\ -1/\sqrt{2} & 0 & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 & -1/2 \\ 0 & -1/\sqrt{2} & 1/2 & -1/2 \end{bmatrix} \text{[2]}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \text{[2]}$$

(d) We have $Q = a^T A a$, where $a = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$. It is standard that with an orthonormal sequence of eigenvectors as above, we have $Q = \sum_i \lambda_i (u_i.a)^2$. In our case $\lambda_1 = \lambda_2 = 0$ so this reduces to

$$Q = 6(u_3.a)^2 - 2(u_4.a)^2 \text{[2]} = (\sqrt{6}(w+x+y+z)/2)^2 - (\sqrt{2}(w+x-y-z)/2)^2 . \text{[1]}$$

We may thus take

$$F = \sqrt{6}(w+x+y+z)/2$$
$$G = \sqrt{2}(w+x-y-z)/2 . \text{[1]}$$

This in turn gives $Q = LM$, where

$$L = F + G = \frac{\sqrt{6}+\sqrt{2}}{2}(w+x) + \frac{\sqrt{6}-\sqrt{2}}{2}(y+z)$$
$$M = F - G = \frac{\sqrt{6}-\sqrt{2}}{2}(w+x) + \frac{\sqrt{6}+\sqrt{2}}{2}(y+z) . \text{[1]}$$