The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2012–13**

Linear Mathematics for Applications **2 hours**

Answer all five questions.

1 You may assume the following row reductions. Some of them are relevant, and some of them are not.

$$\begin{bmatrix} 2 & -1 & -1 & 3 \\ -2 & 2 & 3 & 7 \\ -2 & 12 & 23 & 107 \\ 3 & -2 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & -2 & -2 & 3 \\ -1 & 2 & 12 & -2 \\ -1 & 3 & 23 & -3 \\ 3 & 7 & 107 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 3 \\ 3 & -3 & 1 & 0 & 11 \\ -2 & 2 & 0 & -3 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -2 \\ -1 & -3 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 3 & 11 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Explain what it means for a matrix to be in reduced row echelon form (RREF). *(4 marks)*

(b) Give an example of a 4×4 RREF matrix with pivots in columns 1 and 3, and precisely five nonzero entries. *(2 marks)*

(c) Find the general solution for the following system of linear equations, or prove that there is no solution. *(4 marks)*

$$\begin{aligned} a + d &= b + 3 \\ 3a + c &= 3b + 11 \\ -2a + 2b &= 3d - 7. \end{aligned}$$

(d) Consider the vectors

$$v = \begin{bmatrix} 3 \\ -2 \\ -3 \\ -2 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \\ 7 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -2 \\ 12 \\ 23 \\ 107 \end{bmatrix}$$

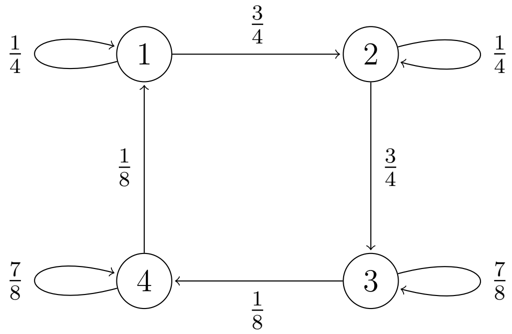
Either express v as a linear combination of u_1, u_2 and u_3, or prove that that is impossible. *(3 marks)*

(e) By performing row operations or otherwise, evaluate the determinant of the following matrix: *(4 marks)*

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 4 & 3 & 0 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(f) Do the columns of A span $\mathbb{R}^4$? Justify your answer. *(3 marks)*

2 Consider the following Markov chain:



(a) Write down the transition matrix P. ***(2 marks)***

(b) Consider the following vectors:

$$u_1 = \begin{bmatrix} 1 \\ -6 \\ 6 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 3 \\ -6 \\ 2 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 2 \\ -6 \\ 3 \end{bmatrix}$$

Show that these are all eigenvectors for P, and find the corresponding eigenvalues. ***(6 marks)***

Note: it is not necessary to calculate the characteristic polynomial.

(c) Using the general theory of Markov chains, write down one more eigenvalue; then find a corresponding eigenvector. ***(6 marks)***

(d) Give an invertible matrix U and a diagonal matrix D such that $P = UDU^{-1}$. Explain how this can be used to calculate P^n. ***(3 marks)***

(e) What is the long run average probability of being in state 1? ***(3 marks)***

3 (a) Are the following statements true or false? Justify your answers. ***(10 marks)***

(i) If V and W are subspaces of $\mathbb{R}^n$, then $\dim(V+W) \leq \dim(V) + \dim(W)$.

(ii) If the list $v_1, \ldots, v_4$ spans $\mathbb{R}^4$, then it is also linearly independent.

(iii) If w_1 can be expressed as a linear combination of w_2, w_3 and w_4, then the list w_1, w_2, w_3, w_4 is linearly independent.

(iv) If $a_1, a_2, a_3, b \in \mathbb{R}^4$ and the matrix $[a_1|a_2|a_3|b]$ row-reduces to the identity matrix, then b is a linear combination of a_1, a_2 and a_3.

(v) If M is a square matrix with $M^T = M$, and u and v are vectors with $u+Mu = v-Mv = 0$, then $u.v = 0$.

(b) Give examples of the following: ***(10 marks)***

(i) A spanning set for $\mathbb{R}^3$ that is not a basis.

(ii) A pair of subspaces $V, W \leq \mathbb{R}^4$ such that $\dim(V) = \dim(W) = 2$ and $\dim(V+W) = 3$.

(iii) A two-dimensional subspace $U \leq \mathbb{R}^4$ such that $w + x + y + z = 0$ for all vectors $\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in U$.

(iv) A non-diagonal matrix whose characteristic polynomial is $t^2 - 1$.

(v) A 2×3 matrix of rank 1 that is not in RREF.

4 Consider the vectors

$$a_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -1 \end{bmatrix} \qquad b_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad b_2 = \begin{bmatrix} -4 \\ 2 \\ -4 \\ 8 \end{bmatrix},$$

and put $V = \operatorname{ann}(a_1, a_2)$, and $W = \operatorname{ann}(b_1, b_2)$.

(a) Find the canonical basis for V. ***(4 marks)***

(b) Find the canonical basis for W. ***(4 marks)***

(c) Find the canonical basis for $V \cap W$. ***(5 marks)***

(d) Find the canonical basis for $V + W$. ***(5 marks)***

(e) Find a vector that lies in $V + W$ but does not lie in V or in W. ***(2 marks)***

5 Consider the matrix $M = \begin{bmatrix} 8 & 2 & 2 \\ 2 & -4 & 5 \\ 2 & 5 & -4 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of M. ***(14 marks)***

(b) Find an orthogonal matrix P and a diagonal matrix D such that $M = PDP^T$. ***(6 marks)***

End of Question Paper