Linear Mathematics for Applications — Exam

(1) You may assume the following row reductions. Some of them are relevant, and some of them are not.

$$\begin{bmatrix} 2 & -1 & -1 & 3 \\ -2 & 2 & 3 & 7 \\ -2 & 12 & 23 & 107 \\ 3 & -2 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & -2 & -2 & 3 \\ -1 & 2 & 12 & -2 \\ -1 & 3 & 23 & -3 \\ 3 & 7 & 107 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 3 \\ 3 & -3 & 1 & 0 & 11 \\ -2 & 2 & 0 & -3 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -2 \\ -1 & -3 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 3 & 11 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Explain what it means for a matrix to be in reduced row echelon form (RREF). **(4 marks)**

(b) Give an example of a 4×4 RREF matrix with pivots in columns 1 and 3, and precisely five nonzero entries. **(2 marks)**

(c) Find the general solution for the following system of linear equations, or prove that there is no solution. **(4 marks)**

$$\begin{aligned} a + d &= b + 3 \\ 3a + c &= 3b + 11 \\ -2a + 2b &= 3d - 7. \end{aligned}$$

(d) Consider the vectors

$$v = \begin{bmatrix} 3 \\ -2 \\ -3 \\ -2 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \\ 7 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -2 \\ 12 \\ 23 \\ 107 \end{bmatrix}$$

Either express v as a linear combination of u_1, u_2 and u_3, or prove that that is impossible. **(3 marks)**

(e) By performing row operations or otherwise, evaluate the determinant of the following matrix: **(4 marks)**

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 4 & 3 & 0 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(f) Do the columns of A span $\mathbb{R}^4$? Justify your answer. **(3 marks)**

Solution:

(a) A matrix A is in RREF if

– Any rows of zeros are at the bottom, after all the nonzero rows. [1]
– In every nonzero row, the first nonzero entry (called the *pivot*) is equal to one. [1]
– Each pivot is further to the right than the pivot in the previous row. [1]
– All entries above or below a pivot are zero. [1]

(b) There are many possible answers: the simplest one is the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{[2]}$$

(c) We write the equations more tidily, convert to an augmented matrix, row reduce (using one of the given row-reductions), and convert back to a system of equations [3].

$$\begin{aligned} a - b + d &= 3 \\ 3a - 3b + c &= 11 \\ -2a + 2b - 3d &= -7 \end{aligned} \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 3 \\ 3 & -3 & 1 & 0 & 11 \\ -2 & 2 & 0 & -3 & -7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \to \begin{aligned} a - b &= 2 \\ c &= 5 \\ d &= 1. \end{aligned}$$

The variable b is independent so we move it to the right hand side. The general solution is $a = b + 2$ with $c = 5$ and $d = 1$ and b arbitrary. [1]

(d) We construct the matrix $[u_1|u_2|u_3|v]$, and observe that it can be reduced by one of the given row-reductions:

$$[u_1|u_2|u_3|v] = \left[\begin{array}{ccc|c} 2 & -2 & -2 & 3 \\ -1 & 2 & 12 & -2 \\ -1 & 3 & 23 & -3 \\ 3 & 7 & 107 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]. \text{[2]}$$

The resulting matrix has a pivot in the last column, which indicates that v cannot be expressed as a linear combination of u_1, u_2 and u_3. [1]

(e) We row-reduce A as follows:

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 4 & 3 & 0 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 6 & 4 \\ 0 & 1 & 3 & 4 \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -8 \\ 0 & 0 & 0 & -4 \\ 0 & 1 & 3 & 4 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -3 & -8 \\ 0 & 0 & 0 & -4 \end{bmatrix} \text{[2]}$$

In step 1 we multiply the first row by $1/2$, so we remember a factor of $1/2$. In step 2 we add multiples of row 1 to rows 2, 3 and 4; this does not contribute any factors to the determinant. In step 3 we add multiples of row 4 to rows 2 and 3; again, this does not contribute any factors. In step 4 we swap the last two rows, then swap the middle two rows. This gives two factors of -1, which cancel each other out. The final matrix is upper triangular, so the determinant is the product of the diagonal entries, which is 12. After dividing by the factor of $1/2$ from the first step, we see that $\det(A) = 24$. [2]

An alternative method is to expand along the top row:

$$\det(A) = 2\det\begin{bmatrix}4 & 3 & 0\\4 & 6 & 4\\2 & 3 & 4\end{bmatrix} - 2\det\begin{bmatrix}2 & 3 & 0\\2 & 6 & 4\\1 & 3 & 4\end{bmatrix}$$

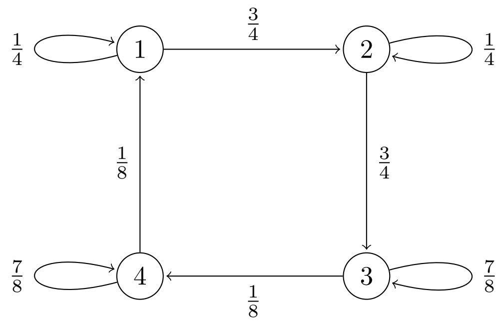
$$\det\begin{bmatrix}4 & 3 & 0\\4 & 6 & 4\\2 & 3 & 4\end{bmatrix} = 4(6\times 4 - 3\times 4) - 3(4\times 4 - 2\times 4) = 24$$

$$\det\begin{bmatrix}2 & 3 & 0\\2 & 6 & 4\\1 & 3 & 4\end{bmatrix} = 2(6\times 4 - 3\times 4) - 3(2\times 4 - 1\times 4) = 12$$

$$\det(A) = 2\times 24 - 2\times 12 = 24.$$

(f) As $\det(A) \neq 0$, we see that A is invertible **[2]**. By a standard theorem, this means that the columns of A form a basis for $\mathbb{R}^4$, so they certainly span $\mathbb{R}^4$.**[1]**

(2) Consider the following Markov chain:



(a) Write down the transition matrix P. **(2 marks)**

(b) Consider the following vectors:

$$u_1 = \begin{bmatrix} 1 \\ -6 \\ 6 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 3 \\ -6 \\ 2 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 2 \\ -6 \\ 3 \end{bmatrix}$$

Show that these are all eigenvectors for P, and find the corresponding eigenvalues. **(6 marks)**

Note: it is not necessary to calculate the characteristic polynomial.

(c) Using the general theory of Markov chains, write down one more eigenvalue; then find a corresponding eigenvector. **(6 marks)**

(d) Give an invertible matrix U and a diagonal matrix D such that $P = UDU^{-1}$. Explain how this can be used to calculate P^n. **(3 marks)**

(e) What is the long run average probability of being in state 1? **(3 marks)**

Solution:

(a) $$P = \begin{bmatrix} 1/4 & 0 & 0 & 1/8 \\ 3/4 & 1/4 & 0 & 0 \\ 0 & 3/4 & 7/8 & 0 \\ 0 & 0 & 1/8 & 7/8 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}. \text{ [2]}$$

(b) We have

$$Pu_1 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ -6 \\ 6 \\ -1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 \\ -6 \\ 6 \\ -1 \end{bmatrix} = \frac{1}{8}u_1 \text{ [1]}$$

$$Pu_2 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -6 \\ 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4 \\ 12 \\ -24 \\ 8 \end{bmatrix} = \frac{1}{2}u_2 \text{ [1]}$$

$$Pu_3 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -6 \\ 3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 \\ 10 \\ -30 \\ 15 \end{bmatrix} = \frac{5}{8}u_3. \text{ [1]}$$

This shows that u_1, u_2 and u_3 are eigenvectors for P, with eigenvalues $\lambda_1 = 1/8$ and $\lambda_2 = 1/2$ and $\lambda_3 = 5/8$. [3]

(c) By the general theory of Markov chains, the matrix P is stochastic, so $\lambda_4 = 1$ is another eigenvalue of P [2]. To find a corresponding eigenvector, we row-reduce the matrix $P - I$:

$$\begin{bmatrix} -3/4 & 0 & 0 & 1/8 \\ 3/4 & -3/4 & 0 & 0 \\ 0 & 3/4 & -1/8 & 0 \\ 0 & 0 & 1/8 & -1/8 \end{bmatrix} \to \begin{bmatrix} -3/4 & 0 & 0 & 1/8 \\ 1 & -1 & 0 & 0 \\ 0 & 3/4 & -1/8 & 0 \\ 0 & 0 & 1/8 & -1/8 \end{bmatrix} \to \begin{bmatrix} 0 & -3/4 & 0 & 1/8 \\ 1 & -1 & 0 & 0 \\ 0 & 3/4 & -1/8 & 0 \\ 0 & 0 & 1/8 & -1/8 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & 0 & -1/6 \\ 1 & -1 & 0 & 0 \\ 0 & 3/4 & -1/8 & 0 \\ 0 & 0 & 1/8 & -1/8 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & -1/6 \\ 1 & 0 & 0 & -1/6 \\ 0 & 0 & -1/8 & 1/8 \\ 0 & 0 & 1/8 & -1/8 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & -1/6 \\ 1 & 0 & 0 & -1/6 \\ 0 & 0 & -1/8 & 1/8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & 0 & -1/6 \\ 1 & 0 & 0 & -1/6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1/6 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{[3]}$$

We deduce that the desired eigenvector has the form $u_4 = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ with $a - d/6 = b - d/6 = c - d = 0$, so $u_4 = \begin{bmatrix} d/6 & d/6 & d & d \end{bmatrix}^T$ with d arbitrary. We choose $d = 6$, giving $u_4 = \begin{bmatrix} 1 & 1 & 6 & 6 \end{bmatrix}^T$. [1]

(d) We now take

$$U = [u_1|u_2|u_3|u_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & 3 & 2 & 1 \\ 6 & -6 & -6 & 6 \\ -1 & 2 & 3 & 6 \end{bmatrix} \text{[1]}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 5/8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \text{[1]}$$

The general theory tells us that $P = UDU^{-1}$. It follows that $P^n = UD^nU^{-1}$, where

$$D^n = \begin{bmatrix} (1/8)^n & 0 & 0 & 0 \\ 0 & (1/2)^n & 0 & 0 \\ 0 & 0 & (5/8)^n & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \text{[1]}$$

(e) The stationary distribution [1]is the eigenvector of eigenvalue 1 where the sum of the entries is 1. As the sum of the entries in u_4 is 14, the stationary distribution is $u_4/14 = \begin{bmatrix} 1/14 & 1/14 & 3/7 & 3/7 \end{bmatrix}^T$ [1]. The long run average probability of being in state 1 is the first entry in the stationary distribution, which is $1/14$ [1].

(3)

(a) Are the following statements true or false? Justify your answers. **(10 marks)**

(i) If V and W are subspaces of $\mathbb{R}^n$, then $\dim(V+W) \leq \dim(V)+\dim(W)$.

(ii) If the list $v_1, \ldots, v_4$ spans $\mathbb{R}^4$, then it is also linearly independent.

(iii) If w_1 can be expressed as a linear combination of w_2, w_3 and w_4, then the list w_1, w_2, w_3, w_4 is linearly independent.

(iv) If $a_1, a_2, a_3, b \in \mathbb{R}^4$ and the matrix $[a_1|a_2|a_3|b]$ row-reduces to the identity matrix, then b is a linear combination of a_1, a_2 and a_3.

(v) If M is a square matrix with $M^T = M$, and u and v are vectors with $u + Mu = v - Mv = 0$, then $u.v = 0$.

(b) Give examples of the following: **(10 marks)**

(i) A spanning set for $\mathbb{R}^3$ that is not a basis.

(ii) A pair of subspaces $V, W \leq \mathbb{R}^4$ such that $\dim(V) = \dim(W) = 2$ and $\dim(V+W) = 3$.

(iii) A two-dimensional subspace $U \leq \mathbb{R}^4$ such that $w + x + y + z = 0$ for all vectors $\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in U$.

(iv) A non-diagonal matrix whose characteristic polynomial is $t^2 - 1$.

(v) A 2×3 matrix of rank 1 that is not in RREF.

Solution:

(a) (i) This is true. [1]The dimension formula says that $\dim(V+W) = \dim(V)+\dim(W) - \dim(V \cap W)$, and this is clearly less than or equal to $\dim(V)+\dim(W)$. [1]

(ii) This is true. [1]In general, if we have a list of n vectors in $\mathbb{R}^n$ (so the number of vectors is the same as the size of each vector) then the list is linearly independent iff it spans iff it is a basis. [1]

(iii) This is false. [1]If w_1 is a linear combination of w_2, w_3 and w_4, then $w_1 = \alpha w_2 + \beta w_3 + \gamma w_4$ for some scalars α, β and γ. This means that we have a nontrivial linear relation $(-1)w_1 + \alpha w_2 + \beta w_3 + \gamma w_4$ on the list w_1, w_2, w_3, w_4, so that list is linearly dependent (not independent). [1]

(iv) This is false. [1]The general method for such problems is as follows: we row-reduce the matrix $[a_1|a_2|a_3|b]$ to get a matrix M; then b is a linear combination of a_1, a_2 and a_3 if and only if M has no pivot in the last column. As the identity matrix has a pivot in the last column, we see that b is *not* a linear combination of a_1, a_2 and a_3. [1]

(v) This is true. [1]The equations $u+Mu = v-Mv = 0$ mean that u and v are eigenvectors of the symmetric matrix M with distinct eigenvalues, namely -1 and $+1$. A general theorem says that such eigenvectors are orthogonal. [1]

(b) In each case, there are many possible correct answers.

(i) The simplest example is the list $e_1, e_2, e_3, 0$. [2]

(ii) The simplest example is to take $V = \mathrm{span}(e_1, e_2)$ and $W = \mathrm{span}(e_1, e_3)$ (so that $V + W = \mathrm{span}(e_1, e_2, e_3)$). [2]

(iii) The simplest example is to take $U = \mathrm{span}(e_1 - e_2, e_3 - e_4)$. [2]

(iv) Possible examples include $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. [2]

(v) Examples include $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. [2]

(4) Consider the vectors

$$a_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -1 \end{bmatrix} \qquad b_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix} \qquad b_2 = \begin{bmatrix} -4 \\ 2 \\ -4 \\ 8 \end{bmatrix},$$

and put $V = \text{ann}(a_1, a_2)$, and $W = \text{ann}(b_1, b_2)$.

(a) Find the canonical basis for V. **(4 marks)**

(b) Find the canonical basis for W. **(4 marks)**

(c) Find the canonical basis for $V \cap W$. **(5 marks)**

(d) Find the canonical basis for $V + W$. **(5 marks)**

(e) Find a vector that lies in $V + W$ but does not lie in V or in W. **(2 marks)**

Solution:

(a) Consider a vector $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$. This lies in V iff $a_1.x = a_2.x = 0$, or equivalently $2x_1 - 2x_4 = x_1 + 2x_2 - 2x_3 - x_4 = 0$. The solution is clearly $x_4 = x_1$ and $x_3 = x_2$ (with x_1 and x_2 arbitrary). Note that we have written higher-numbered variables in terms of lower-numbered ones, as required when finding the canonical basis **[2]**. This gives

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

which shows that the vectors $v_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$ form the canonical basis for V **[2]**.

(b) Similarly, we can write the equations $b_1.x = b_2.x = 0$ with the variables in reverse order as follows:

$$2x_4 - x_3 - x_2 + 2x_1 = 0$$
$$8x_4 - 4x_3 + 2x_2 - 4x_1 = 0.$$

From these equations we deduce

$$6x_2 - 12x_1 = 0$$
$$x_2 = 2x_1$$
$$x_4 = \tfrac{1}{2}x_3 \textbf{[2]},$$

so

$$x = \begin{bmatrix} x_1 \\ 2x_1 \\ x_3 \\ \frac{1}{2}x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/2 \end{bmatrix},$$

so the vectors $w_1 = \begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix}^T$ and $w_2 = \begin{bmatrix} 0 & 0 & 1 & 1/2 \end{bmatrix}^T$ form the canonical basis for W. **[2]**

(c) $V \cap W$ is the set of vectors x satisfying $a_1.x = a_2.x = b_1.x = b_2.x = 0$ [1]. Part (a) tells us that the equations $a_1.x = a_2.x = 0$ are equivalent to $x_4 = x_1$ and $x_3 = x_2$. Part (b) tells us that the equations $b_1.x = b_2.x = 0$ are equivalent to $x_2 = 2x_1$ and $x_4 = \frac{1}{2}x_3$. Putting these together, we get $x_2 = x_3 = 2x_1$ and $x_4 = x_1$ [2], so

$$x = \begin{bmatrix} x_1 \\ 2x_1 \\ 2x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

It follows that the vector $u = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}^T$ is (on its own) the canonical basis for $V \cap W$ [2].

(d) We have

$$V + W = \text{span}(v_1, v_2) + \text{span}(w_1, w_2) = \text{span}(v_1, v_2, w_1, w_2)[1].$$

To find the canonical basis for this space, we row-reduce the matrix $[v_1|v_2|w_1|w_2]^T$ [1]:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} [2].$$

The transposes of the nonzero rows of the last matrix are:

$$t_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad t_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{bmatrix} \qquad t_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/2 \end{bmatrix}.$$

These form the canonical basis for $V + W$. [1]

(e) The vector t_2 lies in $V + W$, but it does not lie in V (because $a_2.t_2 = 1$) and it does not lie in W (because $b_1.t_2 = -2$) [2]. (Note that t_1 is not an example because it lies in V, and t_3 is not an example because it lies in W.)

(5) Consider the matrix $M = \begin{bmatrix} 8 & 2 & 2 \\ 2 & -4 & 5 \\ 2 & 5 & -4 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of M. **(14 marks)**

(b) Find an orthogonal matrix P and a diagonal matrix D such that $M = PDP^T$. **(6 marks)**

Solution:

(a) We first find the characteristic polynomial:

$$\chi_M(t) = \det \begin{bmatrix} 8-t & 2 & 2 \\ 2 & -4-t & 5 \\ 2 & 5 & -4-t \end{bmatrix}$$

$$= (8-t)\det \begin{bmatrix} -4-t & 4 \\ 5 & -4-t \end{bmatrix} - 2\det \begin{bmatrix} 2 & 5 \\ 2 & -4-t \end{bmatrix} + 2\det \begin{bmatrix} 2 & -4-t \\ 2 & 5 \end{bmatrix} \text{[2]}$$

$$\det \begin{bmatrix} -4-t & 5 \\ 5 & -4-t \end{bmatrix} = (-4-t)^2 - 25 = t^2 + 8t - 9$$

$$\det \begin{bmatrix} 2 & 5 \\ 2 & -4-t \end{bmatrix} = -8 - 2t - 10 = -2t - 18$$

$$\det \begin{bmatrix} 2 & -4-t \\ 2 & 5 \end{bmatrix} = 10 - (-8-2t) = 2t + 18$$

$$\chi_M(t) = (8-t)(t^2+8t-9) + 4t + 36 + 4t + 36 = -t^3 + 81t \text{[2]} = -t(t-9)(t+9).$$

Thus, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 9$ and $\lambda_3 = -9$.[1]

To find an eigenvector u_1 of eigenvalue 0, we row-reduce A:

$$\begin{bmatrix} 8 & 2 & 2 \\ 2 & -4 & 5 \\ 2 & 5 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1/4 & 1/4 \\ 2 & -4 & 5 \\ 2 & 5 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1/4 & 1/4 \\ 0 & -9/2 & 9/2 \\ 0 & 9/2 & -9/2 \end{bmatrix} \to \begin{bmatrix} 1 & 1/4 & 1/4 \\ 0 & 1 & -1 \\ 0 & 9/2 & -9/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We conclude that $u_1 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ with $x + z/2 = y - z = 0$. We can take $z = 2$ giving $u_1 = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^T$ [3].

To find an eigenvector u_2 of eigenvalue 9, we row-reduce $A - 9I$:

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -13 & 5 \\ 2 & 5 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -2 \\ 0 & -9 & 9 \\ 0 & 9 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & 9 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We conclude that $u_2 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ with $x - 4z = y - z = 0$. We can take $z = 1$ giving $u_2 = \begin{bmatrix} 4 & 1 & 1 \end{bmatrix}^T$ [3].

To find an eigenvector u_3 of eigenvalue -9, we row-reduce $A + 9I$:

$$\begin{bmatrix} 17 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 17 & 2 & 2 \\ 1 & 5/2 & 5/2 \\ 2 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 0 & -81/2 & -81/2 \\ 1 & 5/2 & 5/2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 1 & 5/2 & 5/2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We conclude that $u_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ with $x = y + z = 0$. We can take $z = -1$ giving $u_3 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ [3].

(b) We now need to find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for M. As M is symmetric and the eigenvalues λ_i are distinct, the eigenvectors u_i are automatically orthogonal. (Alternatively, it is easy to check directly that $u_1.u_2 = u_1.u_3 = u_2.u_3 = 0$.) However, they are not unit vectors, so they do not form an orthonormal basis. Indeed, we have

$$\begin{aligned} \|u_1\| &= \sqrt{1+4+4} = 3 \\ \|u_2\| &= \sqrt{16+1+1} = 3\sqrt{2} \\ \|u_3\| &= \sqrt{0+1+1} = \sqrt{2}\textbf{[2]}. \end{aligned}$$

The vectors $v_i = u_i/\|u_i\|$ form an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for M. We now put

$$P = [v_1|v_2|v_3] = \begin{bmatrix} -1/3 & 2\sqrt{2}/3 & 0 \\ 2/3 & \sqrt{2}/6 & \sqrt{2}/2 \\ 2/3 & \sqrt{2}/6 & -\sqrt{2}/2 \end{bmatrix} \textbf{[2]}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}.\textbf{[2]}$$

The general theory tells us that P is an orthogonal matrix (so $P^{-1} = P^T$) and that $M = PDP^{-1} = PDP^T$.