The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2013–14**

**Linear Mathematics for Applications** **2 hours**

*Answer all five questions.*

**1** You may assume the following row reductions. Some of them are relevant, and some of them are not. Some questions can be done more easily without row-reduction.

$$\begin{bmatrix} 0 & 2 & 0 & -1 & 10 \\ 0 & -1 & 1 & 1 & 5 \\ -1 & -3 & 2 & 2 & -7 \\ -1 & 3 & 3 & 0 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -1 & -1 \\ 2 & -1 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ -1 & 1 & 2 & 0 \\ 10 & 5 & -7 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Give examples of the following: ***(6 marks)***

(i) A $3 \times 3$ RREF matrix $A$ such that $A^T$ is also in RREF.

(ii) A $2 \times 4$ RREF matrix $B$ that is no longer in RREF if you delete the second column.

(iii) A $3 \times 3$ RREF matrix $C$ in which four of the entries are not zero.

(b) Find the general solution for the following system of linear equations, or prove that there is no solution. ***(4 marks)***

$$\begin{aligned} 2b &= 10 + d \\ c + d &= b + 5 \\ 2c + 2d &= a + 3b - 7 \\ 3b + 3c &= a + 36. \end{aligned}$$

(c) Consider the vectors

$$v = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

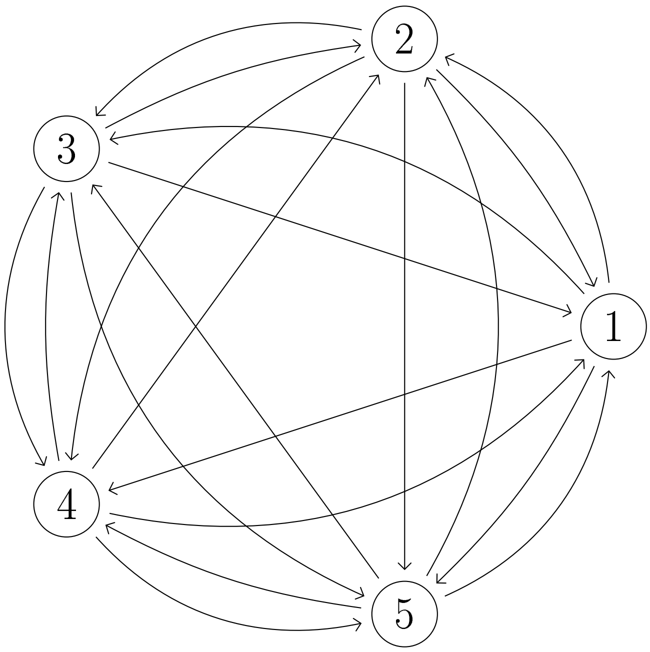
Either express $v$ as a linear combination of $u_1$, $u_2$, $u_3$ and $u_4$, or prove that that is impossible. ***(4 marks)***

(d) Do the vectors $u_i$ in part (c) form a basis for $\mathbb{R}^4$? Justify your answer. ***(2 marks)***

(e) By performing row operations on $C - tI$ or otherwise, evaluate the characteristic polynomial of the following matrix: ***(4 marks)***

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

**2** Consider the following Markov chain:




There is an arrow from every state to every other state, but no arrows from any state to itself. All the arrows have the same probability $p$.

(a) What must $p$ be? ***(2 marks)***

(b) Write down the transition matrix $P$. ***(2 marks)***

(c) Calculate $P^2$, and thus find constants $\alpha$ and $\beta$ such that $P^2 = \alpha P + \beta I_5$. ***(4 marks)***

(d) Show that if $v$ is an eigenvector for $P$ with eigenvalue $\lambda$, then $\lambda^2 = \alpha\lambda + \beta$. ***(2 marks)***

(e) Use (d) to find the eigenvalues of $P$. ***(3 marks)***

(f) You may assume that $P$ has a unique stationary distribution. What is it? ***(3 marks)***

**Hint:** you could use row-reduction, but other methods are much easier.

(g) Find a basis for $\mathbb{R}^5$ consisting of eigenvectors for $P$. ***(4 marks)***

**3** (a) Are the following statements true or false? Justify your answers. ***(10 marks)***

(i) There are subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 4$ and $\dim(V \cap W) = 1$.

(ii) There are subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 4$ and $\dim(V \cap W) = 2$.

(iii) The following list is linearly independent:

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\4\\8\end{bmatrix}, \begin{bmatrix}1\\3\\9\\27\end{bmatrix}, \begin{bmatrix}1\\4\\16\\81\end{bmatrix}, \begin{bmatrix}1\\5\\25\\125\end{bmatrix}.$$

(iv) If $A$ is a $3 \times 3$ matrix with only 2 distinct eigenvalues, then it cannot be diagonalised.

(v) There is a $4 \times 4$ symmetric matrix with characteristic polynomial $t^4 + 1$.

(b) Give examples of the following: ***(10 marks)***

(i) A list $u_1, \ldots, u_4$ of vectors in $\mathbb{R}^2$ such that $u_1, u_2$ is a basis and $u_2, u_3$ is a basis and $u_3, u_4$ is a basis but $u_4, u_1$ is not a basis.

(ii) A pair of subspaces $V, W \leq \mathbb{R}^4$ such that $\dim(V) = \dim(W) = 2$ and

$$V \cap W = \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \mid w + x = x + y = y + z = 0\}.$$

(iii) A non-diagonalisable $3 \times 3$ matrix whose only eigenvalue is 111.

(iv) A stochastic matrix with eigenvalues 1, 1/2 and 1/3.

(v) A $3 \times 2$ matrix of rank 1 that is not in RREF.

**4** Consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad c_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

and the subspaces

$$U = \operatorname{ann}(a_1, a_2) \qquad V = \operatorname{span}(v_1, v_2) \qquad W = \operatorname{ann}(c_1, c_2)$$

in $\mathbb{R}^4$.

(a) Find the canonical basis for $U$. ***(4 marks)***

(b) Find the canonical basis for $V$. ***(3 marks)***

(c) Find the canonical basis for $W$. ***(4 marks)***

(d) Find the canonical basis for $U \cap V \cap W$. ***(5 marks)***

(e) Find the canonical basis for $U + V + W$. ***(4 marks)***

**5** Consider the matrix $A = \frac{1}{27}\begin{bmatrix} 9 & 8 & -8 \\ 8 & 23 & 0 \\ -8 & 0 & -5 \end{bmatrix}$.

(a) State the main results about eigenvalues and eigenvectors of symmetric matrices. ***(4 marks)***

(b) Show that the following are eigenvectors of $A$: ***(2 marks)***

$$u_1 = \begin{bmatrix} 7 \\ -4 \\ -4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}.$$

(c) Find an orthogonal matrix $U$ and a diagonal matrix $D$ such that $A = UDU^T$. ***(9 marks)***

**Hint:** For any square matrix, the sum of the eigenvalues is the same as the sum of the diagonal entries. Because of this, you do not need to calculate the characteristic polynomial.

(d) Find $\lim_{n\to\infty} A^n$. ***(5 marks)***

## End of Question Paper