Linear Mathematics for Applications — Exam

(1) You may assume the following row reductions. Some of them are relevant, and some of them are not. Some questions can be done more easily without row-reduction.

$$\begin{bmatrix} 0 & 2 & 0 & -1 & 10 \\ 0 & -1 & 1 & 1 & 5 \\ -1 & -3 & 2 & 2 & -7 \\ -1 & 3 & 3 & 0 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -1 & -1 \\ 2 & -1 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ -1 & 1 & 2 & 0 \\ 10 & 5 & -7 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Give examples of the following: **(6 marks)**

(i) A 3×3 RREF matrix A such that A^T is also in RREF.

(ii) A 2×4 RREF matrix B that is no longer in RREF if you delete the second column.

(iii) A 3×3 RREF matrix C in which four of the entries are not zero.

(b) Find the general solution for the following system of linear equations, or prove that there is no solution. **(4 marks)**

$$\begin{aligned} 2b &= 10 + d \\ c + d &= b + 5 \\ 2c + 2d &= a + 3b - 7 \\ 3b + 3c &= a + 36. \end{aligned}$$

(c) Consider the vectors

$$v = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Either express v as a linear combination of u_1, u_2, u_3 and u_4, or prove that that is impossible. **(4 marks)**

(d) Do the vectors u_i in part (c) form a basis for $\mathbb{R}^4$? Justify your answer. **(2 marks)**

(e) By performing row operations on $C - tI$ or otherwise, evaluate the characteristic polynomial of the following matrix: **(4 marks)**

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Solution:

(a) (i) The only possibilities are as follows:

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\qquad\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\qquad\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\qquad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Any one is worth two marks. [2]

(ii) The simplest example is

$$B=\begin{bmatrix}0&1&0&0\\0&0&1&0\end{bmatrix}.$$

If we delete the second column, then we get a matrix with a zero row above a nonzero row, which is therefore not in RREF. [2]

(iii) All possible answers have the form

$$C=\begin{bmatrix}1&0&a\\0&1&b\\0&0&0\end{bmatrix}$$

with $a,b\neq 0$. [2]

(b) We write the equations more tidily, convert to an augmented matrix, row reduce (using one of the given row-reductions), and convert back to a system of equations [3].

$$\begin{aligned}0a+2b+0c-d&=10\\0a-b+c+d&=5\\-a-3b+2c+2d&=-7\\-a+3b+3c+0d&=36\end{aligned}\to\left[\begin{array}{cccc|c}0&2&0&-1&10\\0&-1&1&1&5\\-1&-3&2&2&-7\\-1&3&3&0&36\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&0&0&9\\0&1&0&0&8\\0&0&1&0&7\\0&0&0&1&6\end{array}\right]\to\begin{aligned}a&=9\\b&=8\\c&=7\\d&=6.\end{aligned}$$

This gives the unique solution of the original system. [1]

(c) It is easiest to do this by inspection. We have

$$u_2-u_1=\begin{bmatrix}4\\4\\4\\4\end{bmatrix}\qquad\text{and}\qquad v-u_1=\begin{bmatrix}3\\3\\3\\3\end{bmatrix}=\frac{3}{4}(u_2-u_1),$$

which gives

$$v=u_1+\frac{3}{4}(u_2-u_1)=\frac{1}{4}u_1+\frac{3}{4}u_2.\text{[4]}$$

Alternatively, we can row-reduce the matrix $[u_1|u_2|u_3|u_4|v]$:

$$\begin{bmatrix}1&5&8&4&4\\2&6&7&3&5\\3&7&6&2&6\\4&8&5&1&7\end{bmatrix}\to\begin{bmatrix}1&5&8&4&4\\0&-4&-9&-5&-3\\0&-8&-18&-10&-6\\0&-12&-27&-15&-9\end{bmatrix}\to\begin{bmatrix}1&5&8&4&4\\0&1&9/4&5/4&3/4\\0&-8&-18&-10&-6\\0&-12&-27&-15&-9\end{bmatrix}\to\begin{bmatrix}1&0&-13/4&-9/4&1/4\\0&1&9/4&5/4&3/4\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

This shows that $v=\lambda_1u_1+\cdots+\lambda_4u_4$ whenever

$$\begin{aligned}\lambda_1-\tfrac{13}{4}\lambda_3-\tfrac{9}{4}\lambda_4&=\tfrac{1}{4}\\\lambda_2+\tfrac{9}{4}\lambda_3+\tfrac{5}{4}\lambda_4&=\tfrac{3}{4}.\end{aligned}$$

Here λ_3 and λ_4 are independent variables, which can take any value. In particular, we can take $\lambda_3=\lambda_4=0$, giving $\lambda_1=1/4$ and $\lambda_2=3/4$. This gives $v=\frac{1}{4}u_1+\frac{3}{4}u_2$, just as before.

(d) It is easy to see that $u_1+u_3 = u_2+u_4$, or equivalently $u_1-u_2+u_3-u_4 = 0$. This nontrivial relation shows that the list u_1, u_2, u_3, u_4 is linearly dependent and therefore not a basis. [2] Alternatively, we can delete the last column from the row-reduction in (d) to see that the matrix $[u_1|u_2|u_3|u_4]$ does not reduce to I_4, which again proves that the list is not a basis.

(e) We write down $C-tI$ and perform some row operations as follows.

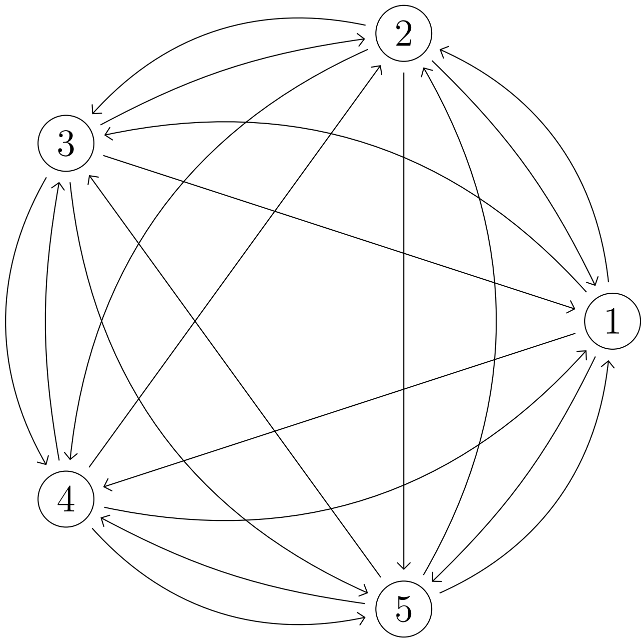
$$\begin{bmatrix} -t & 0 & 1 & 0 \\ 1 & -t & 0 & 1 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{bmatrix} \to \begin{bmatrix} 0 & -t^2 & 1 & t \\ 1 & -t & 0 & 1 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -t^3+1 & t \\ 1 & -t & 0 & 1 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & -t^4+2t \\ 1 & -t & 0 & 1 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{bmatrix} = D[2]$$

We add t times row 2 to row 1, then add t^2 times row 3 to row 1, then add t^3-1 times row 4 to row 1, giving the matrix D shown above. None of these operations change the determinant, so $\det(C-tI) = \det(D)$. We can expand D along the top row to get

$$\det(C-tI) = (-1)^{1+4}(-t^4+2t)\det \begin{bmatrix} 1 & -t & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}.$$

The 3×3 matrix here is upper triangular, so the determinant is the product of the diagonal entries, which is one. It follows that $\det(C-tI) = t^4-2t$. [2]

(2) Consider the following Markov chain:



There is an arrow from every state to every other state, but no arrows from any state to itself. All the arrows have the same probability p.

(a) What must p be? **(2 marks)**

(b) Write down the transition matrix P. **(2 marks)**

(c) Calculate P^2, and thus find constants α and β such that $P^2 = \alpha P + \beta I_5$. **(4 marks)**

(d) Show that if v is an eigenvector for P with eigenvalue λ, then $\lambda^2 = \alpha\lambda + \beta$. **(2 marks)**

(e) Use (d) to find the eigenvalues of P. **(3 marks)**

(f) You may assume that P has a unique stationary distribution. What is it? **(3 marks)**
Hint: you could use row-reduction, but other methods are much easier.

(g) Find a basis for $\mathbb{R}^5$ consisting of eigenvectors for P. **(4 marks)**

Solution:

(a) Each vertex has four outgoing arrows, each with the same probability p. The sum of the outgoing probabilities must be one, so $p = 1/4$. [2]

(b) There are no loops from a vertex back to itself, so the diagonal entries in P are zero. For any two states that are not the same, we have an arrow labelled with $p = 1/4$. Thus, all the off-diagonal entries in P are $1/4$, so

$$P = \frac{1}{4}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}. \text{[2]}$$

(c) We have

$$P^2 = \frac{1}{16}\begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{bmatrix}\begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{bmatrix} = \frac{1}{16}\begin{bmatrix}4&3&3&3&3\\3&4&3&3&3\\3&3&4&3&3\\3&3&3&4&3\\3&3&3&3&4\end{bmatrix}.[2]$$

We want this to be the same as $\alpha P + \beta I$. Looking at the diagonal entries, we see that $4/16 = \beta$, or in other words $\beta = 1/4$ [1]. Looking at the off-diagonal entries, we see that $3/16 = \alpha \times (1/4)$, so $\alpha = 3/4$ [1].

(d) If v is an eigenvector for P with eigenvalue λ, we have

$$\begin{aligned} Pv &= \lambda v \\ P^2 v &= \lambda^2 v \\ (\alpha P + \beta I)v &= \alpha\lambda v + \beta v = (\alpha\lambda + \beta)v. \end{aligned}$$

As $P^2 = \alpha P + \beta I$, it follows that $\lambda^2 v = (\alpha\lambda + \beta)v$, or equivalently $(\lambda^2 - \alpha\lambda - \beta)v = 0$. As v is an eigenvector it is nonzero, so $\lambda^2 - \alpha\lambda - \beta = 0$. [2]

(e) We know that $\alpha = 3/4$ and $\beta = 1/4$, so the equation in (d) becomes $\lambda^2 - 3\lambda/4 - 1/4 = 0$, which factors as $(\lambda - 1)(\lambda + 1/4)$. We see from this that the eigenvalues are 1 and $-1/4$. [3]

(f) It is clear by symmetry that all entries in the stationary distribution must be the same. There are five entries and they must add up to one, so the stationary distribution is

$$u_1 = \begin{bmatrix}1/5 & 1/5 & 1/5 & 1/5 & 1/5\end{bmatrix}.[3]$$

One could also obtain this by row-reducing the matrix $P - I$:

$$\begin{bmatrix}-1&1/4&1/4&1/4&1/4\\1/4&-1&1/4&1/4&1/4\\1/4&1/4&-1&1/4&1/4\\1/4&1/4&1/4&-1&1/4\\1/4&1/4&1/4&1/4&-1\end{bmatrix} \to \begin{bmatrix}1&-1/4&-1/4&-1/4&-1/4\\0&-15/16&5/16&5/16&5/16\\0&5/16&-15/16&5/16&5/16\\0&5/16&5/16&-15/16&5/16\\0&5/16&5/16&5/16&-15/16\end{bmatrix} \to \begin{bmatrix}1&-1/4&-1/4&-1/4&-1/4\\0&1&-1/3&-1/3&-1/3\\0&5/16&-15/16&5/16&5/16\\0&5/16&5/16&-15/16&5/16\\0&5/16&5/16&5/16&-15/16\end{bmatrix} \to$$

$$\begin{bmatrix}1&0&-1/3&-1/3&-1/3\\0&1&-1/3&-1/3&-1/3\\0&0&-5/6&5/12&5/12\\0&0&5/12&-5/6&5/12\\0&0&5/12&5/12&-5/6\end{bmatrix} \to \begin{bmatrix}1&0&-1/3&-1/3&-1/3\\0&1&-1/3&-1/3&-1/3\\0&0&1&-1/2&-1/2\\0&0&5/12&-5/6&5/12\\0&0&5/12&5/12&-5/6\end{bmatrix} \to \begin{bmatrix}1&0&0&-1/2&-1/2\\0&1&0&-1/2&-1/2\\0&0&1&-1/2&-1/2\\0&0&0&-5/8&5/8\\0&0&0&5/8&-5/8\end{bmatrix} \to$$

$$\begin{bmatrix}1&0&0&-1/2&-1/2\\0&1&0&-1/2&-1/2\\0&0&1&-1/2&-1/2\\0&0&0&1&-1\\0&0&0&5/8&-5/8\end{bmatrix} \to \begin{bmatrix}1&0&0&0&-1\\0&1&0&0&-1\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&0\end{bmatrix}$$

From this last matrix we see again that all entries in the stationary distribution must be the same.

(g) The stationary distribution u_1 is an eigenvector, of eigenvalue 1 [1]. The eigenvectors of eigenvalue $-1/4$ must satisfy $(P + \frac{1}{4}I)u = 0$. However, every entry in $P + \frac{1}{4}I$ is $1/4$, so a vector $u = \begin{bmatrix}v & w & x & y & z\end{bmatrix}^T$ satisfies $(P + \frac{1}{4}I)u = 0$ if and only if $(v + w + x + y + z)/4 = 0$ [1]. We therefore have linearly independent eigenvectors as follows:

$$u_2 = \begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix} \quad u_3 = \begin{bmatrix}0\\1\\0\\0\\-1\end{bmatrix} \quad u_4 = \begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix} \quad u_5 = \begin{bmatrix}0\\0\\0\\1\\-1\end{bmatrix}.$$

The full list $u_1, \ldots, u_5$ is now a basis for $\mathbb{R}^5$ consisting of eigenvectors for P. [2]

(3)

(a) Are the following statements true or false? Justify your answers. **(10 marks)**

(i) There are subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 4$ and $\dim(V \cap W) = 1$.

(ii) There are subspaces $V, W \leq \mathbb{R}^6$ with $\dim(V) = \dim(W) = 4$ and $\dim(V \cap W) = 2$.

(iii) The following list is linearly independent:

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\\4\\8\end{bmatrix}, \quad \begin{bmatrix}1\\3\\9\\27\end{bmatrix}, \quad \begin{bmatrix}1\\4\\16\\81\end{bmatrix}, \quad \begin{bmatrix}1\\5\\25\\125\end{bmatrix}.$$

(iv) If A is a 3×3 matrix with only 2 distinct eigenvalues, then it cannot be diagonalised.

(v) There is a 4×4 symmetric matrix with characteristic polynomial $t^4 + 1$.

(b) Give examples of the following: **(10 marks)**

(i) A list $u_1, \ldots, u_4$ of vectors in $\mathbb{R}^2$ such that u_1, u_2 is a basis and u_2, u_3 is a basis and u_3, u_4 is a basis but u_4, u_1 is not a basis.

(ii) A pair of subspaces $V, W \leq \mathbb{R}^4$ such that $\dim(V) = \dim(W) = 2$ and

$$V \cap W = \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \mid w + x = x + y = y + z = 0\}.$$

(iii) A non-diagonalisable 3×3 matrix whose only eigenvalue is 111.

(iv) A stochastic matrix with eigenvalues 1, 1/2 and 1/3.

(v) A 3×2 matrix of rank 1 that is not in RREF.

Solution:

(a) (i) This is false [1]. The dimension formula says that $\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$, which would give $\dim(V + W) = 4 + 4 - 1 = 7$, but it is impossible for a subspace of $\mathbb{R}^6$ to have dimension 7. [1]

(ii) This is true. [1]As an example, we could have $V = \text{span}(e_1, e_2, e_3, e_4)$ and $W = \text{span}(e_1, e_2, e_5, e_6)$ so $V \cap W = \text{span}(e_1, e_2)$. [1]

(iii) This is false. [1]No list of 5 vectors in $\mathbb{R}^4$ can be linearly independent. [1]

(iv) This is false. [1]The matrix

$$A = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

has only two eigenvalues (namely 0 and 1), but it is diagonal and so is certainly diagonalisable. [1]

(v) This is false. [1]Any symmetric matrix has real eigenvalues, but the roots of $t^4 + 1$ are not real. [1]

(b) In each case, there are many possible correct answers.

(i) Note that a list v, w in $\mathbb{R}^2$ is a basis iff v and w are nonzero, and not multiples of each other. Given this, we see that the list $e_1, e_2, e_1 + e_2, e_1$ is as required. [2]

(ii) Put $u_1 = \begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}$, so

$$\{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \mid w + x = x + y = y + z = 0\} = \text{span}(u_1).$$

Then the spaces $V = \text{span}(u_1, e_1)$ and $W = \text{span}(u_1, e_2)$ have dimension two with $V \cap W = \text{span}(u_1)$ as required. [2]

(iii) The standard example is the Jordan block $\begin{bmatrix} 111 & 1 & 0 \\ 0 & 111 & 1 \\ 0 & 0 & 111 \end{bmatrix}$. **[2]**

(iv) The obvious example is $\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$. **[2]**

(v) One example is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ **[2]**

(4) Consider the vectors

$$a_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad a_2 = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix} \quad v_1 = \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\0\\0\\-1\end{bmatrix} \quad c_1 = \begin{bmatrix}1\\-1\\1\\1\end{bmatrix} \quad c_2 = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$

and the subspaces

$$U = \operatorname{ann}(a_1, a_2) \qquad V = \operatorname{span}(v_1, v_2) \qquad W = \operatorname{ann}(c_1, c_2)$$

in $\mathbb{R}^4$.

(a) Find the canonical basis for U. **(4 marks)**

(b) Find the canonical basis for V. **(3 marks)**

(c) Find the canonical basis for W. **(4 marks)**

(d) Find the canonical basis for $U \cap V \cap W$. **(5 marks)**

(e) Find the canonical basis for $U + V + W$. **(4 marks)**

Solution:

(a) For a vector $u = \begin{bmatrix}p & q & r & s\end{bmatrix}^T$ to lie in U, it must satisfy $u.a_1 = u.a_2 = 0$, or equivalently $p + q = r - s = 0$ [1]. This means that u must have the form

$$u = \begin{bmatrix}p\\-p\\r\\r\end{bmatrix} = p\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + q\begin{bmatrix}0\\0\\1\\1\end{bmatrix}.[2]$$

From this it follows that the vectors

$$u_1 = \begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}^T \qquad \text{and} \qquad u_2 = \begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix}^T$$

form the canonical basis for U. [1]

(b) We use the following row-reduction:

$$\left[\begin{array}{c} v_1^T \\ \hline v_2^T \end{array}\right] [1] = \begin{bmatrix}1 & -1 & -1 & -1\\1 & 0 & 0 & -1\end{bmatrix} \to \begin{bmatrix}1 & -1 & -1 & -1\\0 & 1 & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & 1 & 0\end{bmatrix} [1]$$

It follows that the vectors $v_1' = \begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}^T$ and $v_2' = \begin{bmatrix}0 & 1 & 1 & 0\end{bmatrix}^T$ form the canonical basis for V. [1]

(c) For a vector $w = \begin{bmatrix}p & q & r & s\end{bmatrix}^T$ to lie in U, it must satisfy $w.c_1 = w.c_2 = 0$, or equivalently $p - q + r + s = 0$ and $p + q + r - s = 0$. [1]By adding and subtracting these equations, we obtain $p + r = 0$ and $q - s = 0$, so

$$w = \begin{bmatrix}p\\q\\-p\\q\end{bmatrix} = p\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + q\begin{bmatrix}0\\1\\0\\1\end{bmatrix}.[2]$$

It follows that the vectors

$$w_1 = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} \qquad \text{and } w_2 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

form the canonical basis for W. [1]

(d) From (b) it is clear that V is the set of vectors of the form $v = \begin{bmatrix} w & x & x & -w \end{bmatrix}^T$.[2] For v to lie in $U \cap V \cap W$ it must also satisfy $v.a_1 = v.a_2 = v.c_1 = v.c_2 = 0$, or in other words

$$w + x = x - (-w) = w - x + x + (-w) = w + x + x - (-w) = 0.\text{[1]}$$

This just reduces to $x = -w$ [1], so

$$v = \begin{bmatrix} w & -w & -w & -w \end{bmatrix}^T = w \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix}^T = w\, v_1.$$

We conclude that the vector $v_1 = \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix}^T$ (on its own) is the canonical basis for $U \cap V \cap W$. [1]

(e) The space $U + V + W$ is spanned by $u_1, u_2, v_1', v_2', w_1$ and w_2. [2] We form a matrix with these vectors as rows, and row-reduce it:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{[1]}$$

From this we conclude that $U + V + W$ is all of $\mathbb{R}^4$, so the canonical basis is e_1, e_2, e_3, e_4. [1]Alternatively, we can observe that the vector e_3 is $u_2 + v_2' - w_2$, so it lies in $U + V + W$. It follows in turn that the vectors $e_1 = w_1 + e_3$ and $e_2 = v_2' - e_3$ and $e_4 = u_2 - e_3$ also lie in $U + V + W$, so again we see that $U + V + W$ is all of $\mathbb{R}^4$.

(5) Consider the matrix $A = \frac{1}{27}\begin{bmatrix} 9 & 8 & -8 \\ 8 & 23 & 0 \\ -8 & 0 & -5 \end{bmatrix}$.

(a) State the main results about eigenvalues and eigenvectors of symmetric matrices. **(4 marks)**

(b) Show that the following are eigenvectors of A: **(2 marks)**

$$u_1 = \begin{bmatrix} 7 \\ -4 \\ -4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}.$$

(c) Find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^T$. **(9 marks)** **Hint:** For any square matrix, the sum of the eigenvalues is the same as the sum of the diagonal entries. Because of this, you do not need to calculate the characteristic polynomial.

(d) Find $\lim_{n\to\infty} A^n$. **(5 marks)**

Solution:

(a) For any real symmetric matrix, all eigenvalues are real [2], and eigenvectors corresponding to different eigenvectors are orthogonal [2]. (For the second part, two marks will also be given for saying that there is an orthonormal basis of eigenvectors.)

(b) We have

$$Au_1 = \frac{1}{27}\begin{bmatrix} 9 & 8 & -8 \\ 8 & 23 & 0 \\ -8 & 0 & -5 \end{bmatrix}\begin{bmatrix} 7 \\ -4 \\ -4 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 63 \\ -36 \\ 36 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ -4 \\ 4 \end{bmatrix} = \frac{1}{3}u_1$$
$$Au_2 = \frac{1}{27}\begin{bmatrix} 9 & 8 & -8 \\ 8 & 23 & 0 \\ -8 & 0 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -36 \\ 9 \\ -72 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -4 \\ 1 \\ -8 \end{bmatrix} = -\frac{1}{3}u_2.$$

This shows that u_1 is an eigenvector of eigenvalue $\lambda_1 = 1/3$, and u_2 is an eigenvector of eigenalue $\lambda_2 = -1/3$. [2]

(c) If we let λ_3 denote the third eigenvalue, then the hint tells us that

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{27}(9 + 23 - 5) = 1.$$

As $\lambda_1 = 1/3$ and $\lambda_2 = -1/3$, this just gives $\lambda_3 = 1$ [2]. To find the corresponding eigenvector u_3, we row-reduce the matrix $A - I$:

$$\begin{bmatrix} -2/3 & 8/27 & -8/27 \\ 8/27 & -4/27 & 0 \\ -8/27 & 0 & -32/27 \end{bmatrix} \to \begin{bmatrix} 1 & -4/9 & 4/9 \\ 8/27 & -4/27 & 0 \\ -8/27 & 0 & -32/27 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & -4/9 & 4/9 \\ 0 & -4/243 & -32/243 \\ 0 & -32/243 & -256/243 \end{bmatrix} \to \begin{bmatrix} 1 & -4/9 & 4/9 \\ 0 & 1 & 8 \\ 0 & -32/243 & -256/243 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{bmatrix} \text{ [2]}$$

This tells us that u_3 must have the form $\begin{bmatrix} x & y & z \end{bmatrix}^T$ with $x + 4z = y + 8z = 0$, so $u_3 = z\begin{bmatrix} -4 & -8 & 1 \end{bmatrix}^T$ for some arbitrary nonzero scalar z. We choose $z = -1$ giving $u_3 = \begin{bmatrix} 4 & 8 & -1 \end{bmatrix}^T$. [1]

Next, as A is symmmetric and the eigenvectors u_k have different eigenvalues, they are automatically orthogonal. (It is also not hard to check that directly, by calculating that $u_1.u_2 = u_1.u_3 = u_2.u_3 = 0$.) However, they are not orthonormal, because

$$\begin{aligned} \|u_1\|^2 &= 7^2 + (-4)^2 + (-4)^2 = 81 \\ \|u_2\|^2 &= 4^2 + (-1)^2 + 8^2 = 81 \\ \|u_3\|^2 &= 4^2 + 8^2 + (-1)^2 = 81, \end{aligned}$$

so $\|u_1\| = \|u_2\| = \|u_3\| = \sqrt{81} = 9$ [2]. It follows that the vectors $u_k/9$ form an orthonormal basis. We therefore obtain an orthogonal diagonalisation $A = UDU^T$ with

$$U = \left[\ \tfrac{u_1}{9} \ \middle|\ \tfrac{u_2}{9} \ \middle|\ \tfrac{u_3}{9}\ \right] = \frac{1}{9}\begin{bmatrix} 7 & 4 & 4 \\ -4 & -1 & 8 \\ -4 & 8 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{[2]}$$

Recall also that U is an orthogonal matrix, so $U^{-1} = U^T$.

(d) We now have $A^n = UD^nU^T$ for all $n \geq 0$ [1]. Now put

$$D^\infty = \lim_{n\to\infty} D^n = \lim_{n\to\infty} \begin{bmatrix} (1/3)^n & 0 & 0 \\ 0 & (-1/3)^n & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{[2]}$$

It follows that

$$\begin{aligned} \lim_{n\to\infty} A^n &= UD^\infty U^T \text{[1]} \\ &= \frac{1}{81}\begin{bmatrix} 7 & 4 & 4 \\ -4 & -1 & 8 \\ -4 & 8 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & -4 & -4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{bmatrix} \\ &= \frac{1}{81}\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 8 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 7 & -4 & -4 \\ 4 & -1 & 8 \\ 4 & 8 & -1 \end{bmatrix} = \frac{1}{81}\begin{bmatrix} 16 & 32 & -4 \\ 32 & 64 & -8 \\ -4 & -8 & 1 \end{bmatrix} \text{[1]} \end{aligned}$$