The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Linear Mathematics for Applications – Mock exam 1 **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. ***(3 marks)***

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Row-reduce the following matrix. ***(6 marks)***

$$D = \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 3 & 3 & 3 & 5 & 7 \\ 5 & 5 & 5 & 8 & 9 \end{bmatrix}$$

(c) You may assume the row-reduction

$$\begin{bmatrix} 1 & -2 & 0 & 1 & -3 & 2 \\ -2 & 4 & 0 & -2 & 6 & -4 \\ 1 & -2 & -1 & 5 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the general solution of the system of equations

$$\begin{aligned} v - 2w + y - 3z &= 2 \\ -2v + 4w - 2y + 6z &= -4 \\ v - 2w - x + 5y - 4z &= 0 \end{aligned}$$

Then find a specific solution (with no free variables) where $x = 0$. ***(6 marks)***

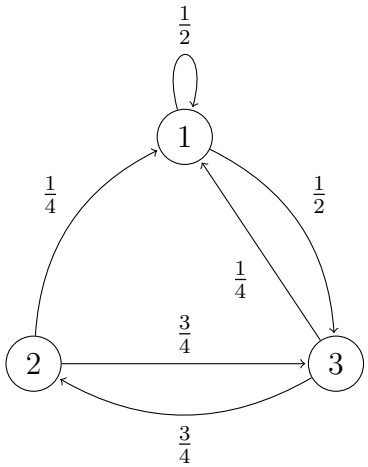
(d) Find the determinant of the following matrix: ***(3 marks)***

$$E = \begin{bmatrix} 0 & a & 0 & 0 \\ b & c & d & e \\ f & g & 0 & h \\ i & j & 0 & k \end{bmatrix}$$

(e) State, with justification, which of the following matrices are invertible. ***(7 marks)***

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix} \qquad J = \begin{bmatrix} 100 & 20 & 3 & 123 \\ 300 & 10 & 7 & 317 \\ 500 & 70 & 1 & 571 \\ 200 & 60 & 9 & 269 \end{bmatrix}$$

2 Consider the following Markov chain:



(a) Write down the associated transition matrix P. ***(2 marks)***

(b) You may assume that the eigenvalues of P are $\lambda_1 = 1$ and $\lambda_2 = -3/4$ and $\lambda_3 = 1/4$. Find corresponding eigenvectors u_1, u_2 and u_3. ***(9 marks)***

(c) Find a stationary distribution for the system. ***(2 marks)***

(d) Show that the vector $v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ can be expressed as a linear combination of u_1 and u_2. ***(4 marks)***

(e) At time $t = 0$ the system has an equal probability of being in any of the three states. What is the probability of being in state 1 at time $t = 10$? ***(8 marks)***

3 (a) Are the following statements true or false? Justify your answers. ***(9 marks)***

(i) No list of four vectors in $\mathbb{R}^5$ can span $\mathbb{R}^5$.

(ii) Every linearly independent list is a basis.

(iii) Let u and v be eigenvectors for a square matrix A, with different eigenvalues; then u and v are orthogonal.

(iv) Let V and W be subspaces of $\mathbb{R}^n$ with $V \cap W = \{0\}$; then $\dim(V + W) = \dim(V) + \dim(W)$.

(b) Which of the following sets is a subspace of $\mathbb{R}^4$? Justify your answers. ***(9 marks)***

$$V_1 = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w^2 - x^2 + y^2 - z^2 = 0\}$$
$$V_2 = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid 2w - 3x + 4y - 5z = 0\}$$
$$V_3 = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid 2w - 3x + 4y - 5z = 1\}$$
$$V_4 = \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + x^2 + y^3 + z^4 = 0\}.$$

(c) Give examples of the following. ***(7 marks)***

(i) A list of four vectors in $\mathbb{R}^2$ such that any two of them form a basis.

(ii) A list of four vectors in $\mathbb{R}^4$ such that any two of them are linearly independent, but any three of them are linearly dependent.

(iii) A 3×3 matrix that has only two distinct eigenvalues.

4 Let V be the set of all vectors in $\mathbb{R}^5$ of the form

$$x = \begin{bmatrix} p & p+q & 0 & -p-q & -p \end{bmatrix}^T$$

(where $p, q \in \mathbb{R}$ are arbitrary). Also, put

$$W = \{x \in \mathbb{R}^5 \mid x_1 + x_2 + x_3 + 3x_4 = x_1 + x_2 + x_3 - x_4 + 2x_5 = 0\}.$$

(a) Find the canonical basis for V. ***(5 marks)***

(b) Find the canonical basis for W. ***(6 marks)***

(c) Find the canonical basis for $V + W$. ***(5 marks)***

(d) Write down a formula relating the dimensions of V, W, $V + W$ and $V \cap W$, and use it to determine $\dim(V \cap W)$. ***(3 marks)***

(e) Find a basis for $V \cap W$ ***(6 marks)***

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5 Put

$$A = \begin{bmatrix} 9 & 6 & 2 & 3 \\ 6 & 0 & 0 & 2 \\ 2 & 0 & 0 & 6 \\ 3 & 2 & 6 & 9 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) Show that $\det(A) = 1024 = 2^{10}$. ***(4 marks)***

(b) Show that the vectors u_i are eigenvectors for A, and determine the corresponding eigenvalues. ***(6 marks)***

Note: You do not need to find the characteristic polynomial or perform any row-reduction.

(c) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. ***(6 marks)***

(d) Express the quadratic form

$$Q = 9(w^2 + z^2) + 12(wx + yz) + 6wz + 4(wy + xz)$$

as $Q = F^2 + G^2 - H^2 - J^2$, where F, G, H and J are linear forms. ***(6 marks)***

(e) What are the rank and signature of Q? ***(3 marks)***

End of Question Paper