Linear Mathematics for Applications — Mock Exam 1

(1)

(a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. **(3 marks)**

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Row-reduce the following matrix. **(6 marks)**

$$D = \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 3 & 3 & 3 & 5 & 7 \\ 5 & 5 & 5 & 8 & 9 \end{bmatrix}$$

(c) You may assume the row-reduction

$$\begin{bmatrix} 1 & -2 & 0 & 1 & -3 & 2 \\ -2 & 4 & 0 & -2 & 6 & -4 \\ 1 & -2 & -1 & 5 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the general solution of the system of equations

$$\begin{aligned} v - 2w + y - 3z &= 2 \\ -2v + 4w - 2y + 6z &= -4 \\ v - 2w - x + 5y - 4z &= 0 \end{aligned}$$

Then find a specific solution (with no free variables) where $x = 0$. **(6 marks)**

(d) Find the determinant of the following matrix: **(3 marks)**

$$E = \begin{bmatrix} 0 & a & 0 & 0 \\ b & c & d & e \\ f & g & 0 & h \\ i & j & 0 & k \end{bmatrix}$$

(e) State, with justification, which of the following matrices are invertible. **(7 marks)**

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix} \qquad J = \begin{bmatrix} 100 & 20 & 3 & 123 \\ 300 & 10 & 7 & 317 \\ 500 & 70 & 1 & 571 \\ 200 & 60 & 9 & 269 \end{bmatrix}$$

Solution:

(a) The matrix A is in RREF [1]. The matrix B is not in RREF, because it has a nonzero entry above the pivot in the last column [1]. The matrix C is not in RREF because the pivot in the second row is to the left of the pivot in the first row [1].

(b)

$$\begin{bmatrix}2&2&2&4&6\\3&3&3&5&7\\5&5&5&8&9\end{bmatrix}\to\begin{bmatrix}1&1&1&2&3\\3&3&3&5&7\\5&5&5&8&9\end{bmatrix}\to\begin{bmatrix}1&1&1&2&3\\0&0&0&-1&-2\\0&0&0&-2&-6\end{bmatrix}\to\begin{bmatrix}1&1&1&2&3\\0&0&0&1&2\\0&0&0&-2&-6\end{bmatrix}$$

$$\to\begin{bmatrix}1&1&1&0&-1\\0&0&0&1&2\\0&0&0&0&-2\end{bmatrix}\to\begin{bmatrix}1&1&1&0&-1\\0&0&0&1&2\\0&0&0&0&1\end{bmatrix}\to\begin{bmatrix}1&1&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\ [6]$$

(c) The given system of equations corresponds to the augmented matrix

$$A=\left[\begin{array}{ccccc|c}1&-2&0&1&-3&2\\-2&4&0&-2&6&-4\\1&-2&-1&5&-4&0\end{array}\right].[1]$$

We are told that this row-reduces to the matrix

$$A'=\left[\begin{array}{ccccc|c}1&-2&0&1&-3&2\\0&0&1&-4&1&2\\0&0&0&0&0&0\end{array}\right],$$

so our original system is equivalent to the system

$$\begin{aligned}v-2w+y-3z&=2\\x-4y+z&=2.[2]\end{aligned}$$

The general solution is $v=2+2w-y+3z$ and $x=2+4y-z$ with w, y and z arbitrary [1]. To find a specific solution with $x=0$ we can take $w=y=0$ and $z=2$; this gives $v=2+2w-y+3z=8$ and $x=2+4y-z=0$ so $\begin{bmatrix}v&w&x&y&z\end{bmatrix}=\begin{bmatrix}8&0&0&0&2\end{bmatrix}$ [2].

(d) Expand along the first row, then down the middle column:

$$\det(E)=-a\det\begin{bmatrix}b&d&e\\f&0&h\\i&0&k\end{bmatrix}=(-a)(-d)\det\begin{bmatrix}f&h\\i&k\end{bmatrix}=ad(fk-hi)=adfk-adhi.[3]$$

(e) The matrix F can be row-reduced to the identity:

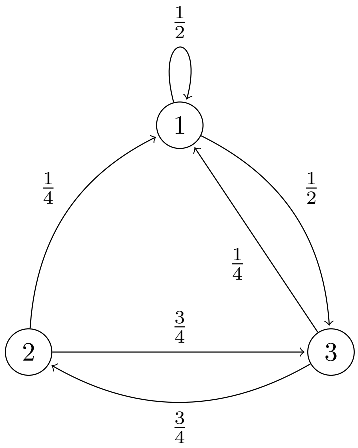
$$\begin{bmatrix}1&1&1&1\\1&1&1&2\\1&1&2&2\\1&2&2&2\end{bmatrix}\to\begin{bmatrix}1&1&1&1\\0&0&0&1\\0&0&1&1\\0&1&1&1\end{bmatrix}\to\begin{bmatrix}1&1&1&0\\0&0&0&1\\0&0&1&0\\0&1&1&0\end{bmatrix}\to\begin{bmatrix}1&1&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

It follows that F is invertible [2]. The matrix G is not square and so cannot be invertible [1]. Next, we can evaluate $\det(H)$ by expanding down the first column:

$$\det(H)=-3\det\begin{bmatrix}2&2\\-1&-1\end{bmatrix}-\det\begin{bmatrix}1&-2\\-1&-1\end{bmatrix}=-3\times0-(-1-2)=3.$$

As $\det(H)\neq0$, we see that H is invertible.[2] In the matrix J the last column is the sum of the first three columns, so the columns are linearly dependent, so J is not invertible. [2]

(2) Consider the following Markov chain:



(a) Write down the associated transition matrix P. **(2 marks)**

(b) You may assume that the eigenvalues of P are $\lambda_1 = 1$ and $\lambda_2 = -3/4$ and $\lambda_3 = 1/4$. Find corresponding eigenvectors u_1, u_2 and u_3. **(9 marks)**

(c) Find a stationary distribution for the system. **(2 marks)**

(d) Show that the vector $v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ can be expressed as a linear combination of u_1 and u_2. **(4 marks)**

(e) At time $t = 0$ the system has an equal probability of being in any of the three states. What is the probability of being in state 1 at time $t = 10$? **(8 marks)**

Solution:

(a) $P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 0 & 3/4 \\ 1/2 & 3/4 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$. [2]

(b) To find an eigenvector of eigenvalue 1 we row-reduce $P - I$:

$$\begin{bmatrix} -1/2 & 1/4 & 1/4 \\ 0 & -1 & 3/4 \\ 1/2 & 3/4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -3/4 \\ 1/2 & 3/4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -3/4 \\ 0 & 1 & -3/4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -7/8 \\ 0 & 1 & -3/4 \\ 0 & 0 & 0 \end{bmatrix} \text{ [2]}$$

Any eigenvector $u_1 = \begin{bmatrix} x & y & z \end{bmatrix}^T$ of eigenvalue 1 must therefore satisfy $x - (7/8)z = y - (3/4)z = 0$, so $u_1 = \begin{bmatrix} (7/8)z & (3/4)z & z \end{bmatrix}^T$ with z arbitrary [1]. It will be convenient to take $z = 8$ giving $u_1 = \begin{bmatrix} 7 & 6 & 8 \end{bmatrix}^T$.

To find an eigenvector u_2 of eigenvalue $-3/4$ we row-reduce $P + \frac{3}{4}I$:

$$\begin{bmatrix} 5/4 & 1/4 & 1/4 \\ 0 & 3/4 & 3/4 \\ 1/2 & 3/4 & 3/4 \end{bmatrix} \to \begin{bmatrix} 1 & 1/5 & 1/5 \\ 0 & 1 & 1 \\ 1/2 & 3/4 & 3/4 \end{bmatrix} \to \begin{bmatrix} 1 & 1/5 & 1/5 \\ 0 & 1 & 1 \\ 0 & 13/20 & 13/20 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ [2]}$$

From this we see that we can take $u_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ [1]. Similarly, to find an eigenvector u_2 of eigenvalue $-3/4$ we row-reduce $P - \frac{1}{4}I$:

$$\begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 0 & -1/4 & 3/4 \\ 1/2 & 3/4 & -1/4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 1/2 & 3/4 & -1/4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 1/4 & -3/4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \text{ [2]}$$

From this we see that we can take $u_3 = \begin{bmatrix} -4 & 3 & 1 \end{bmatrix}^T$ [1].

(c) A stationary distribution q must be an eigenvector of eigenvalue 1, so it must be a multiple of u_1 [1], say $q = \begin{bmatrix} 7t & 6t & 8t \end{bmatrix}^T$. For a probability vector the sum of the entries must be equal to one, which means that $21t = 1$ so $t = 1/21$ and $q = \begin{bmatrix} 1/3 & 2/7 & 8/21 \end{bmatrix}^T$ [1].

(d) By inspection we have $u_1 + u_2 = \begin{bmatrix} 7 & 7 & 7 \end{bmatrix}^T$ so $v = (u_1 + u_2)/7$. For a more systematic approach, we need to find α_1, α_2 and α_3 such that $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = v$, or equivalently

$$\alpha_1 \begin{bmatrix} 7 \\ 6 \\ 8 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{aligned} 7\alpha_1 - 4\alpha_3 &= 1 \\ 6\alpha_1 + \alpha_2 + 3\alpha_3 &= 1 \\ 8\alpha_1 - \alpha_2 + \alpha_3 &= 1 \end{aligned} \qquad [2]$$

These equations are easily solved to give $\alpha_1 = \alpha_2 = 1/7$ and $\alpha_3 = 0$ so $v = (u_1 + u_2)/7$ as before [2].

(e) We are told that at $t = 0$ the three states all have equal probability, which must be $1/3$, so the initial distribution is

$$r_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{v}{3} = \frac{u_1 + u_2}{21}. [2]$$

It follows that

$$r_{10} = P^{10} r_0 [1] = \tfrac{1}{21}(P^{10}u_1 + P^{10}u_2) = \tfrac{1}{21}\left(u_1 + (-\tfrac{3}{4})^{10} u_2\right) [2] = \begin{bmatrix} 7/21 \\ 6/21 + (-3/4)^{10}/21 \\ 8/21 - (-3/4)^{10}/21 \end{bmatrix} [1].$$

The probability of being in state 1 at $t = 10$ is the first entry in r_{10}, which is just $7/21 = 1/3$ [2].

(3)

(a) Are the following statements true or false? Justify your answers. **(9 marks)**

(i) No list of four vectors in $\mathbb{R}^5$ can span $\mathbb{R}^5$.

(ii) Every linearly independent list is a basis.

(iii) Let u and v be eigenvectors for a square matrix A, with different eigenvalues; then u and v are orthogonal.

(iv) Let V and W be subspaces of $\mathbb{R}^n$ with $V \cap W = \{0\}$; then $\dim(V + W) = \dim(V) + \dim(W)$.

(b) Which of the following sets is a subspace of $\mathbb{R}^4$? Justify your answers. **(9 marks)**

$$\begin{aligned} V_1 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w^2 - x^2 + y^2 - z^2 = 0\} \\ V_2 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid 2w - 3x + 4y - 5z = 0\} \\ V_3 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid 2w - 3x + 4y - 5z = 1\} \\ V_4 &= \{\begin{bmatrix} w & x & y & z \end{bmatrix}^T \in \mathbb{R}^4 \mid w + x^2 + y^3 + z^4 = 0\}. \end{aligned}$$

(c) Give examples of the following. **(7 marks)**

(i) A list of four vectors in $\mathbb{R}^2$ such that any two of them form a basis.

(ii) A list of four vectors in $\mathbb{R}^4$ such that any two of them are linearly independent, but any three of them are linearly dependent.

(iii) A 3×3 matrix that has only two distinct eigenvalues.

Solution:

(a) (i) This is true [1]. A general theorem in the notes says that any list that spans $\mathbb{R}^n$ must have at least n elements [1].

(ii) This is not true [1]. For example, the vectors $u_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ and $u_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$ give a linearly independent list u_1, u_2 in $\mathbb{R}^3$, but this list does not span $\mathbb{R}^3$ and so is not a basis. [1]

(iii) This is false [1]. For a concrete counterexample, consider the matrix $A = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$ and the vectors $u = \begin{bmatrix}1\\ 0\end{bmatrix}$ and $v = \begin{bmatrix}1\\ 1\end{bmatrix}$. These are eigenvectors of eigenvalue 0 and 1 respectively, but they are not orthogonal. The statement is true for symmetric matrices, but not more generally [2].

(iv) This is true [1]. It is a general theorem that for any subspaces $V, W \leq \mathbb{R}^n$ we have $\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W)$. In the case where $V \cap W = \{0\}$ we have $\dim(V \cap W) = 0$ and so $\dim(V + W) = \dim(V) + \dim(W)$ [1].

(b)
- The set V_1 is not a subspace [1]. Indeed, the vectors $a = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$ and $a' = \begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}^T$ are elements of V_1 but the vector $a + a' = \begin{bmatrix}2 & 0 & 0 & 0\end{bmatrix}$ is not; so V_1 is not closed under addition and cannot be a subspace. [1]
- The set V_2 is a subspace [1]. Indeed, it is clear that the zero vector lies in V_2. Next, suppose we have two elements $a = \begin{bmatrix}w & x & y & z\end{bmatrix}^T$ and $a' = \begin{bmatrix}w' & x' & y' & z'\end{bmatrix}^T$ in V_2, so $2w - 3x + 4y - 5z = 0$ and $2w' - 3x' + 4y' - 5z' = 0$. By adding these equations, we see that $2(w + w') - 3(x + x') + 4(y + y') - 5(z + z') = 0$, so the vector $a + a' = \begin{bmatrix}w + w' & x + x' & y + y' & z + z'\end{bmatrix}$ also lies in V_2. This shows that V_2 is closed under addition. Similarly, for any $t \in \mathbb{R}$ we have $2tw - 3tx + 4ty - 5tz = 0$, showing that $ta \in V_2$. This means that V_2 is closed under scalar multiplication and so is a subspace [2].
- The set V_3 is not a subspace [1], because it does not contain the zero vector [1].
- The set V_4 is not a subspace [1]. Indeed, the vector $a = \begin{bmatrix}-1 & 1 & 0 & 0\end{bmatrix}^T$ is an element of V_4, but the vector $-a = \begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}^T$ is not, which shows that V_4 is not closed under scalar multiplication. [1].

(c) (i) The simplest example is the list $\begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 1\end{bmatrix}, \begin{bmatrix}1\\ 2\end{bmatrix}, \begin{bmatrix}1\\ 3\end{bmatrix}$ [2].

(ii) The simplest example is the list $\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 2\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 3\\ 0\end{bmatrix}$ [3].

(iii) The simplest example is the matrix $A = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$ [2].

(4) Let V be the set of all vectors in $\mathbb{R}^5$ of the form

$$x = \begin{bmatrix}p & p + q & 0 & -p - q & -p\end{bmatrix}^T$$

(where $p, q \in \mathbb{R}$ are arbitrary). Also, put

$$W = \{x \in \mathbb{R}^5 \mid x_1 + x_2 + x_3 + 3x_4 = x_1 + x_2 + x_3 - x_4 + 2x_5 = 0\}.$$

(a) Find the canonical basis for V. **(5 marks)**

(b) Find the canonical basis for W. **(6 marks)**

(c) Find the canonical basis for $V+W$. **(5 marks)**

(d) Write down a formula relating the dimensions of V, W, $V+W$ and $V\cap W$, and use it to determine $\dim(V\cap W)$. **(3 marks)**

(e) Find a basis for $V\cap W$ **(6 marks)**.

Solution:

(a) The general element of V can be written as $x=pu_1+qu_2$, where

$$u_1=\begin{bmatrix}1 & 1 & 0 & -1 & -1\end{bmatrix}^T \qquad u_2=\begin{bmatrix}0 & 1 & 0 & -1 & 0\end{bmatrix}^T,$$

so $V=\text{span}(u_1,u_2)$ [2]. Using the row-reduction

$$\left[\begin{array}{c}u_1^T\\ \hline u_2^T\end{array}\right]=\begin{bmatrix}1 & 1 & 0 & -1 & -1\\ 0 & 1 & 0 & -1 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & -1 & 0\end{bmatrix}\ [2]$$

we see that the vectors

$$v_1=\begin{bmatrix}1 & 0 & 0 & 0 & -1\end{bmatrix}^T \qquad \text{and} \qquad v_2=\begin{bmatrix}0 & 1 & 0 & -1 & 0\end{bmatrix}^T$$

form the canonical basis for V [1].

(b) To find the canonical basis for W we write the defining equations with the variables in descending order, and find the solution in a form that expresses higher-numbered variables in terms of lower-numbered ones. The equations are

$$3x_4+x_3+x_2+x_1=0$$
$$2x_5-x_4+x_3+x_2+x_1=0.[1]$$

The first equation gives $x_4=-\frac{1}{3}x_3-\frac{1}{3}x_2-\frac{1}{3}x_1$. On the other hand, we can subtract the two equation to get $2x_5-4x_4=0$ or in other words $x_5=2x_4=-\frac{2}{3}x_3-\frac{2}{3}x_2-\frac{2}{3}x_1$ [2]. We now have

$$x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\\-\frac{1}{3}x_3-\frac{1}{3}x_2-\frac{1}{3}x_1\\-\frac{2}{3}x_3-\frac{2}{3}x_2-\frac{2}{3}x_1\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\\-1/3\\-2/3\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\\-1/3\\-2/3\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\\-1/3\\-2/3\end{bmatrix}\ [2].$$

It follows that the vectors

$$w_1=\begin{bmatrix}1\\0\\0\\-1/3\\-2/3\end{bmatrix} \quad \text{and} \quad w_2=\begin{bmatrix}0\\1\\0\\-1/3\\-2/3\end{bmatrix} \quad \text{and} \quad w_3=\begin{bmatrix}0\\0\\1\\-1/3\\-2/3\end{bmatrix}$$

form the canonical basis for W [1].

(c) It now follows that $V+W=\text{span}(v_1,v_2,w_1,w_2,w_3)$ [1]. To make this canonical we perform the following row-reduction:

$$\left[\begin{array}{c}v_1^T\\ \hline v_2^T\\ \hline w_1^T\\ \hline w_2^T\\ \hline w_3^T\end{array}\right][1]=\begin{bmatrix}1 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1/3 & -2/3\\ 0 & 1 & 0 & -1/3 & -2/3\\ 0 & 0 & 1 & -1/3 & -2/3\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & -1/3 & 1/3\\ 0 & 1 & 0 & -1/3 & -2/3\\ 0 & 0 & 1 & -1/3 & -2/3\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & -1/3 & 1/3\\ 0 & 0 & 0 & 1/3 & -1/3\\ 0 & 0 & 1 & -1/3 & -2/3\end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & -2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{[2]}$$

This shows that the canonical basis for $V + W$ consists of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \text{[1]}.$$

(d) The general dimension formula says that $\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W)$ [1], so $\dim(V \cap W) = \dim(V) + \dim(W) - \dim(V + W)$. The dimension of a subspace is just the number of vectors in the canonical basis, so $\dim(V) = 2$ and $\dim(W) = 3$ and $\dim(V + W) = 4$ [1]. This gives $\dim(V \cap W) = 2 + 3 - 4 = 1$ [1].

(e) This problem can be done in several different ways, but the following is probably the easiest. Any element $x \in V \cap W$ must have the form

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T = \begin{bmatrix} p & p+q & 0 & -p-q & -p \end{bmatrix}^T$$

and must satisfy

$$x_1 + x_2 + x_3 + 3x_4 = x_1 + x_2 + x_3 - x_4 + 2x_5 = 0.\text{[2]}$$

This reduces to

$$p + (p + q) + 0 + 3(-p - q) = p + (p + q) + 0 - (-p - q) + 2(-p) = 0$$

or in other words $-p - 2q = p + 2q = 0$, so $p = -2q$ [2]. This in turn gives

$$x = \begin{bmatrix} p \\ p+q \\ 0 \\ -p-q \\ -p \end{bmatrix} = \begin{bmatrix} -2q \\ -q \\ 0 \\ q \\ 2q \end{bmatrix} = q \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \text{[1]}.$$

This shows that the vector $\begin{bmatrix} -2 & -1 & 0 & 1 & 2 \end{bmatrix}$ is (on its own) a basis for $V \cap W$ [1]. (To get the canonical basis, we would divide this vector by -2.)

(5) Put

$$A = \begin{bmatrix} 9 & 6 & 2 & 3 \\ 6 & 0 & 0 & 2 \\ 2 & 0 & 0 & 6 \\ 3 & 2 & 6 & 9 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) Show that $\det(A) = 1024 = 2^{10}$. **(4 marks)**

(b) Show that the vectors u_i are eigenvectors for A, and determine the corresponding eigenvalues. **(6 marks)**
Note: You do not need to find the characteristic polynomial or perform any row-reduction.

(c) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. **(6 marks)**

(d) Express the quadratic form

$$Q = 9(w^2 + z^2) + 12(wx + yz) + 6wz + 4(wy + xz)$$

as $Q = F^2 + G^2 - H^2 - J^2$, where F, G, H and J are linear forms. **(6 marks)**

(e) What are the rank and signature of Q? **(3 marks)**

Solution:

(a) We expand the determinant along the second row. The 6 at the beginning of the row is in the $(2,1)$ slot and so has a sign $(-1)^{2+1} = -1$. The 2 at the end of the row is in the $(2,4)$ slot and so has a sign $(-1)^{2+4} = +1$. This gives

$$\det(A) = -6 \det \begin{bmatrix} 6 & 2 & 3 \\ 0 & 0 & 6 \\ 2 & 6 & 9 \end{bmatrix} + 2 \det \begin{bmatrix} 9 & 6 & 2 \\ 2 & 0 & 0 \\ 3 & 2 & 6 \end{bmatrix}. [1]$$

Each of the above 3×3 determinants can be expanded along the middle row:

$$\det \begin{bmatrix} 6 & 2 & 3 \\ 0 & 0 & 6 \\ 2 & 6 & 9 \end{bmatrix} = -6 \det \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} = -6 \times 32 = -192 [1]$$

$$\det \begin{bmatrix} 9 & 6 & 2 \\ 2 & 0 & 0 \\ 3 & 2 & 6 \end{bmatrix} = -2 \det \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} = -2 \times 32 = -64. [1]$$

Putting this together we get

$$\det(A) = -6 \times (-192) + 2 \times (-64) = 1024. [1]$$

(b) We have

$$Au_1 = \begin{bmatrix} 9 & 6 & 2 & 3 \\ 6 & 0 & 0 & 2 \\ 2 & 0 & 0 & 6 \\ 3 & 2 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -4 \\ 2 \end{bmatrix} = -2u_1$$

$$Au_2 = \begin{bmatrix} 9 & 6 & 2 & 3 \\ 6 & 0 & 0 & 2 \\ 2 & 0 & 0 & 6 \\ 3 & 2 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 8 \\ -4 \end{bmatrix} = -4u_2$$

$$Au_3 = \begin{bmatrix} 9 & 6 & 2 & 3 \\ 6 & 0 & 0 & 2 \\ 2 & 0 & 0 & 6 \\ 3 & 2 & 6 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \\ -8 \\ -16 \end{bmatrix} = 8u_3$$

$$Au_4 = \begin{bmatrix} 9 & 6 & 2 & 3 \\ 6 & 0 & 0 & 2 \\ 2 & 0 & 0 & 6 \\ 3 & 2 & 6 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 32 \\ 16 \\ 16 \\ 32 \end{bmatrix} = 16u_4 [4]$$

From this we see that the vectors u_i are eigenvctors for A, with eigenvalues $\lambda_1 = -2$, $\lambda_2 = -4$, $\lambda_3 = 8$ and $\lambda_4 = 16$ [2].

(c) We need to find an orthonormal basis for $\mathbb{R}^4$ consisting of eigenvectors for A [1]. The matrix A is symmetric, and the vectors u_i are eigenvectors for A with distinct eigenvalues, so the general theory tells us that they are all orthogonal to each other. (Alternatively, we can just check that directly $u_i.u_j = 0$ for all $i \neq j$.) [1]However, we do not have an orthonormal basis because u_i is not a unit vector. Instead, we have $u_1.u_1 = 1^2 + (-2)^2 + 2^2 + (-1)^2 = 10$ [1], and similar calculations show that $u_2.u_2 = u_3.u_3 = u_4.u_4 = 10$ as well. It follows that if we put $v_i = u_i/\sqrt{10}$ then the list v_1, v_2, v_3, v_4 is an orthonormal basis as required [1].

The general theory now tells us that we can take

$$P = \left[\begin{array}{c|c|c|c} v_1 & v_2 & v_3 & v_4 \end{array}\right] = \begin{bmatrix} 1/\sqrt{10} & 1/\sqrt{10} & 2/\sqrt{10} & 2/\sqrt{10} \\ -2/\sqrt{10} & -2/\sqrt{10} & 1/\sqrt{10} & 1/\sqrt{10} \\ 2/\sqrt{10} & -2/\sqrt{10} & -1/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 1/\sqrt{10} & -2/\sqrt{10} & 2/\sqrt{10} \end{bmatrix} [1]$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix} [1]$$

(d) We have $Q = a^T A a$, where $a = \begin{bmatrix} w & x & y & z \end{bmatrix}^T$ [1]. It is standard that with an orthonormal sequence of eigenvectors as above, we have $Q = \sum_i \lambda_i (v_i.a)^2$ [1]. In our case this gives

$$\begin{aligned} Q &= -2(a.v_1)^2 - 4(a.v_2)^2 + 8(a.v_3)^2 + 16(a.v_4)^2 [1] \\ &= (a.\sqrt{8}v_3)^2 + (a.4v_4)^2 - (a.\sqrt{2}v_1)^2 - (a.2v_2)^2 [1]. \end{aligned}$$

In other words, we have $Q = F^2 + G^2 - H^2 - J^2$, where

$$\begin{aligned} F &= a.\sqrt{8}v_3 = 4w/\sqrt{5} + 2x/\sqrt{5} - 2y/\sqrt{5} - 4z/\sqrt{5} \\ G &= a.4v_4 = 8w/\sqrt{10} + 4x/\sqrt{10} + 4y/\sqrt{10} + 8z/\sqrt{10} \\ H &= a.\sqrt{2}v_1 = w/\sqrt{5} - 2x/\sqrt{5} + 2y/\sqrt{5} - z/\sqrt{5} \\ J &= a.2v_2 = 2w/\sqrt{10} - 4x/\sqrt{10} - 4y/\sqrt{10} + 2z/\sqrt{10} [2] \end{aligned}$$

(e) The rank is the number of nonzero eigenvalues, which is 4 [1]. The signature is the number of positive eigenvalues minus the number of negative eigenvalues, which is 0 [2].