The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2011–12**

Linear Mathematics for Applications – Mock exam 2 **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

1 (a) Which of the following matrices are in reduced row echelon form (RREF)? Explain your answers. ***(4 marks)***

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Row-reduce the following matrix. ***(6 marks)***

$$E = \begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 3 & 3 & 3 & 5 & 7 \\ 5 & 5 & 5 & 8 & 9 \end{bmatrix}$$

(c) Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 2 \end{bmatrix} \quad v_5 = \begin{bmatrix} 7 \\ 6 \\ 4 \\ 5 \end{bmatrix} \quad v_6 = \begin{bmatrix} 7 \\ 4 \\ 6 \\ 5 \end{bmatrix}$$

You may assume the row-reduction

$$\begin{bmatrix} 1 & 4 & 2 & 1 & 7 & 7 \\ 3 & 1 & 1 & 2 & 6 & 4 \\ 7 & 0 & 1 & 5 & 4 & 6 \\ 2 & 2 & 1 & 2 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(i) Either express v_5 as a linear combination of v_1, v_2, v_3, v_4, or explain why that is not possible. ***(3 marks)***

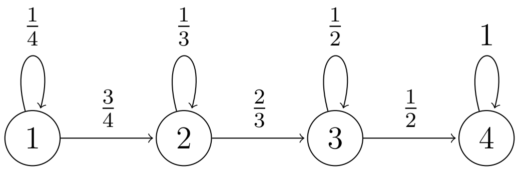
(ii) Either express v_6 as a linear combination of v_1, v_2, v_3, v_4, or explain why that is not possible. ***(3 marks)***

(d) Find inverses for the matrices F, G and H below, and explain why J does not have an inverse. ***(9 marks)***

$$F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 100 & 10 & 111 \\ 10 & 1 & 100 & 111 \\ 10 & 100 & 1 & 111 \\ 100 & 1 & 10 & 111 \end{bmatrix}$$

Hint: for H, trial and error may be easier than a systematic method.

2 Consider the following Markov chain:



(a) Write down the associated transition matrix P. *(2 marks)*

(b) Show that the following vectors are eigenvectors for P, and find the corresponding eigenvalues. *(4 marks)*

$$u_1 = \begin{bmatrix} 1 \\ -9 \\ 24 \\ -16 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ 3 \end{bmatrix}$$

(c) Find all the other eigenvalues and eigenvectors. *(9 marks)*

(e) At time $t = 0$ the system is in state 1. What is the probability that it is in state 4 at $t = 5$? *(10 marks)*

3 (a) Are the following statements true or false? You do not need to justify your answers. ***(9 marks)***

(i) Let A be a 3×5 matrix with linearly independent rows. Then the rank of A is 5.

(ii) Let A be a 5×5 matrix with $A^T = A$, and let u and v be vectors with $Au = 2u$ and $Av = 3v$. Then $u.v = 0$.

(iii) Let U be a 4×4 matrix whose columns are all orthogonal to each other. Then $U^T U = I$.

(iv) The list $\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}7\\1\end{bmatrix}, \begin{bmatrix}4\\4\end{bmatrix}$ is linearly independent.

(v) The list $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ spans $\mathbb{R}^4$.

(vi) Let A be a 5×5 matrix with $\text{rank}(A) = 5$. Then A is invertible.

(vii) Let $u_1, u_2, u_3, u_4, u_5, u_6$ be a linearly independent list of vectors in $\mathbb{R}^6$. Then the list u_1, u_3, u_5 is also linearly independent.

(viii) Let $u_1, u_2, u_3, u_4, u_5, u_6$ be a list of vectors that spans $\mathbb{R}^6$. Then the list u_1, u_3, u_5 also spans $\mathbb{R}^6$.

(ix) Every subspace of $\mathbb{R}^3$ is either a line or a plane.

(b) Which of the following sets is a subspace of $\mathbb{R}^4$? Here you do need to justify your answers. ***(8 marks)***

$$\begin{aligned}
V_1 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w \geq x\} \\
V_2 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w = x = y = z\} \\
V_3 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid w + x + y + z \text{ is an integer }\} \\
V_4 &= \{\begin{bmatrix}w & x & y & z\end{bmatrix}^T \in \mathbb{R}^4 \mid wxyz = 0\}.
\end{aligned}$$

(c) Give examples of the following. ***(8 marks)***

(i) A linearly dependent list that spans $\mathbb{R}^2$.

(ii) A 2×2 matrix that is invertible but not orthogonal.

(iii) A subspace of $\mathbb{R}^3$ that contains the vector $\begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ and has dimension 2.

(iv) A 4×4 matrix A such that $A \neq I_4$ and $\chi_A(t) = (t-1)^4$.

4 Consider the vectors

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ -6 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 3 \\ 6 \\ 0 \\ -6 \end{bmatrix} \qquad p = \begin{bmatrix} 11 \\ 22 \\ 22 \\ 0 \end{bmatrix} \qquad q = \begin{bmatrix} 222 \\ -111 \\ 0 \\ 0 \end{bmatrix}$$

Put $V = \text{span}(a_1, a_2, a_3)$ and $W = \text{ann}(a_1, a_2, a_3)$.

(a) Find the canonical basis for V. ***(4 marks)***

(b) Find the canonical basis for W. ***(8 marks)***

(c) Find the canonical basis for $V + W$. ***(5 marks)***

(d) Use the dimension formula to determine $V \cap W$. ***(4 marks)***

(e) Prove that $p \in V$ and $q \in W$. ***(4 marks)***

5 Put

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix}.$$

You may assume the following row-reductions. (Some of them are useful, and some of them are not.)

$$\begin{bmatrix} 2 & 3 & 7 \\ 3 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 7 & 3 & 7 \\ 3 & 7 & 3 \\ 7 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 9 & 3 & 7 \\ 3 & 9 & 3 \\ 7 & 3 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 7 \\ 3 & -2 & 3 \\ 7 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -7 & 3 & 7 \\ 3 & -7 & 3 \\ 7 & 3 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -9 & 3 & 7 \\ 3 & -9 & 3 \\ 7 & 3 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Find the eigenvalues of A. ***(6 marks)***

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. ***(10 marks)***

(c) Express the quadratic form

$$Q = 2x^2 + 2y^2 + 2z^2 + 6xy + 6yz + 14xz$$

as $Q = L^2 - M^2$, where L and M are linear forms. Hence find linear forms F and G such that $Q = FG$. ***(7 marks)***

(d) What are the rank and signature of Q? ***(2 marks)***

End of Question Paper