Lecture 2

(Reduced) row-echelon form

Definition 5.1: Let A be a matrix of real numbers. Recall that A is said to be in *reduced row-echelon form* (RREF) if the following hold:

- **RREF0**: Any rows of zeros come at the of the matrix, after all the nonzero rows.
- **RREF1**: In any nonzero row, the first nonzero entry is equal to . These entries are called *pivots*.
- **RREF2**: In any nonzero row, the pivot is further to the right than the pivots in all previous rows.
- **RREF3**: If a column contains a pivot, then all other entries in that column are .

We will also say that a system of linear equations (in a specified list of variables) is in RREF if the corresponding augmented matrix is in RREF.

If RREF0, RREF1 and RREF2 are satisfied but not RREF3 then we say that A is in (unreduced) row-echelon form.

(Reduced) row-echelon form

Example 5.2:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & \mathbf{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathbf{1} & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & \mathbf{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A is not in RREF because the middle row is zero and the bottom row is not (RREF0 fails).The matrix B is also not in RREF because the first nonzero entry in the top row is 2 rather than 1 (RREF1 fails). The matrix C is not in RREF because the pivot in the bottom row is to the left of the pivots in the previous rows (RREF2 fails). The matrix D is not in RREF because the last column contains a pivot and also another nonzero entry (RREF3 fails). On the other hand, the matrix

$$E = \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & \mathbf{1} & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF.

RREF for systems of equations

Example 5.3: The system of equations

$$\begin{aligned} x - z &= 1 \\ y &= 2 \end{aligned}$$

is in RREF because its augmented matrix is in RREF:

$$A = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

The system of equations

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 2 \\ z &= 3 \end{aligned}$$

is not in RREF because its augmented matrix is not in RREF:

$$B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solving RREF systems

If a system of equations is in RREF, it can be solved very easily.

$$\begin{aligned} w + 2x + 3z &= 10 \\ y + 4z &= 20. \end{aligned} \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 10 \\ 0 & 0 & 1 & 4 & 20 \end{array}\right]$$

Variables in non-pivot columns are *independent*; they can take any value, and we move them to the right hand side. Variables in pivot columns are *dependent*; they stay on the left. The equations now express the dependent variables in terms of the independent ones.

$$\begin{aligned} w &= 10 - 2x - 3z \\ y &= 20 - 4z \end{aligned}$$

Sometimes it is convenient to introduce new symbols for the independent variables, say λ and μ. Then the solution is

$$\begin{aligned} w &= 10 - 2\lambda - 3\mu \\ x &= \lambda \\ y &= 20 - 4\mu \\ z &= \mu \end{aligned}$$

where λ and μ can take arbitrary values.

Solving RREF systems — degenerate cases

The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 & 13 \end{array}\right]$$

has a pivot in every column to the left of the bar, so there are no independent variables. It corresponds to the system

$$w = 10 \qquad x = 11 \qquad y = 12 \qquad z = 13$$

which is its own (unique) solution.
The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

has a pivot in the last column, to the right of the bar. It corresponds to the system

$$\begin{aligned} w + z &= 0 \\ 0 &= 1 \end{aligned} \qquad \begin{aligned} x + y &= 0 \\ 0 &= 0 \end{aligned}$$

so there is clearly no solution.

Row operations

Let A be a matrix. The following operations on A are called *elementary row operations*:

ERO1: Exchange two rows.

ERO2: Multiply a row by a nonzero constant.

ERO3: Add a multiple of one row to another row.

We write $A \to B$ if A can be converted to B by a sequence of EROs. As all EROs are reversible, we see that if $A \to B$ then also $B \to A$.

Theorem

Let A be a matrix.

(a) *By applying a sequence of row operations to A, one can obtain a matrix B that is in RREF.*

(b) *Although there are various different sequences that reduce A to RREF, they all give the same matrix B at the end of the process.*

In a moment we will recall the standard procedure for row-reduction. It is not hard to prove (by induction on the number of rows) that this procedure always works as advertised, so (a) is true. Statement (b) is an important fact but we will not prove it in this course.

Row reduction

To reduce a matrix A to RREF, we do the following.

(a) If all rows are zero, then A is already in RREF, so we are done.

(b) Otherwise, we find a row that has a nonzero entry as far to the left as possible. Let this entry be u, in the k'th column of the j'th row say. Because we went as far to the left as possible, all entries in columns 1 to $k-1$ of the matrix are zero.

(c) We now exchange the first row with the j'th row (which does nothing if j happens to be equal to one).

(d) Next, we multiply the first row by u^{-1}. We now have a 1 in the k'th column of the first row.

(e) We now subtract multiples of the first row from all the other rows to ensure that the k'th column contains nothing except for the pivot in the first row.

(f) We now ignore the first row and apply row operations to the remaining rows to put them in RREF.

(g) If we put the first row back in, we have a matrix that is nearly in RREF, except that the first row may have nonzero entries above the pivots in the lower rows. This can easily be fixed by subtracting multiples of those lower rows.

Row reduction example

Consider the following sequence of reductions:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ \quad & \quad & \quad & \quad & \quad \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & -2 & -1 & -13 \\ \quad & \quad & \quad & \quad & \quad \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & -5/2 & -25/2 \end{bmatrix} \xrightarrow{6}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 13/2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{7} \begin{bmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{8} \begin{bmatrix} 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{9} \begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Exchange the first two rows; Multiply the first row by $\quad$; Add the first row to the third row; Divide the second row by $\quad$; Subtract the second row from $\quad$; Multiply the third row by $-2/5$; Subtract half the bottom row from the middle row; Subtract the middle row from the top row; Add the bottom row to the top row.

Row reduction example

Consider the following sequence of reductions:

$$C = \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ -1 & -1 & 3 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 4 & 5 & 4 \\ 1 & 2 & 1 & 7 & 6 & 8 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 1 & -3 & 7 & 7 & 4 \\ 0 & 0 & 4 & 4 & 4 & 8 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & -2 & -2 & 1 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & -3 & 4 & 4 & 1 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & -5 & -5 & -5 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & -3 & 3 & 2 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{6}$$

$$\begin{bmatrix} 1 & 0 & -3 & -3 & -4 & -6 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{7} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{8} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add row 1 to rows 2 and 4, and subtract it from rows 3 and 5; Subtract row 2 from row $\quad$; Exchange rows 3 and $\quad$; Add $\quad$ times row 3 to row 4, and subtract $\quad$ times row 3 from row 5; Divide row 4 by $\quad$, then add 5 times row 4 to row 5; Subtract 2 times row 2 from row $\quad$; Add 3 times row 3 to row 1; Subtract 3 times row 4 from row 2, and subtract row 4 from row 3.

Deleting columns

We previously saw the following row-reduction:

$$\begin{bmatrix} 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & -1 & 1 & -2 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & -1 & 1 & -2 \\ 0 & 0 & -2 & -1 & -13 \\ -1 & -2 & 0 & -1 & -8 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

We can delete the middle column and it still works the same way:

$$\begin{bmatrix} 0 & 0 & & -1 & -13 \\ -1 & -2 & & 1 & -2 \\ -1 & -2 & & -1 & -8 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & & 1 & -2 \\ 0 & 0 & & -1 & -13 \\ -1 & -2 & & -1 & -8 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & & 0 & 3 \\ 0 & 0 & & 0 & 4 \\ 0 & 0 & & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 & -13 \\ -1 & -2 & 1 & -2 \\ -1 & -2 & -1 & -8 \end{bmatrix} \to \begin{bmatrix} -1 & -2 & 1 & -2 \\ 0 & 0 & -1 & -13 \\ -1 & -2 & -1 & -8 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

(However, the final result is no longer in RREF; we need further row operations to fix that.)

In general: suppose that $A \to A'$, and that B is obtained by deleting some columns from A, and that B' is obtained by deleting the corresponding columns from A'. Then $B \to B'$.

Solution by row-reduction

Theorem 6.8: Let A be an augmented matrix, and let A' be obtained from A by a sequence of row operations. Then the system of equations corresponding to A has the same solutions (if any) as the system of equations corresponding to A'.

This should be fairly clear. The three types of elementary row operations correspond to reordering our system of equations, multiplying both sides of one equation by a nonzero constant, and adding one equation to another one. None of these operations changes the solution set. We thus have the following method:

Method 6.9: To solve a system of linear equations:

(a) Write down the corresponding augmented matrix.
(b) Row-reduce it to RREF
(c) Convert it back to a new system of equations, which will have exactly the same solutions as the old ones.
(d) Read off the solutions (which is easy for a system in RREF).

Example solution by row-reduction

We will try to solve the following system:

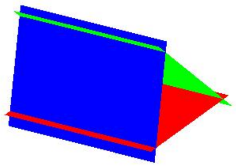
$$
\begin{array}{ccccccc}
2x & + & y & + & z & = & 1 \\
4x & + & 2y & + & 3z & = & -1 \\
6x & + & 3y & - & z & = & 11
\end{array}
$$

We construct and then row-reduce the augmented matrix:

$$
\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 2 & 3 & -1 \\ & & & \end{array}\right]
\xrightarrow{1}
\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -4 & 8 \end{array}\right]
\xrightarrow{2}
\left[\begin{array}{ccc|c} 2 & 1 & 1 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \end{array}\right]
\xrightarrow{3}
\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{array}\right]
$$

There is a pivot in the rightmost column, which means that there are no solutions for the original system.

Each of the equations defines a plane. These are arranged like the three faces of a Toblerone packet, so there is no point where they all meet.



Example solution by row-reduction

We will solve the equations

$$
\begin{aligned}
a + b + c + d &= 4 \\
a + b - c - d &= 0 \\
a - b + c - d &= 0 \\
a - b - c + d &= 0.
\end{aligned}
$$

The corresponding augmented matrix can be row-reduced as follows:

$$
\left[\begin{array}{cccc|c} 1 & 1 & 1 & & 4 \\ 1 & 1 & -1 & & 0 \\ 1 & -1 & 1 & & 0 \\ 1 & -1 & -1 & & 0 \end{array}\right]
\xrightarrow{1}
\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & -2 & -2 & -4 \\ 1 & -1 & 1 & -1 & 0 \\ & & & & \end{array}\right]
\xrightarrow{2}
\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ & & & & \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]
\xrightarrow{3}
$$

$$
\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]
\xrightarrow{4}
\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -2 & 0 & 0 & -2 \\ & & & & \end{array}\right]
\xrightarrow{5}
\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]
\xrightarrow{6}
$$

$$
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]
\xrightarrow{7}
\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ & & & & \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]
$$

Subtract row 1 from row 2, and row 3 from row 4; Multiply rows 2 and 4 by $-1/2$; Subtract row 2 from row 1, and row 4 from row 3; Subtract row 1 from row 3, and row 2 from row 4; Multiply rows 3 and 4 by $-1/2$; Subtract row 3 from row 1, and row 4 from row 2; Exchange rows 2 and 3.
The final matrix corresponds to the equations $a = 1$, $b = 1$, $c = 1$ and $d = 1$, which give the unique solution to the original system of equations.

Homogeneous systems

A system of linear equations is *homogeneous* if the values on the right hand side are all zero. Example:

$$
\begin{aligned}
a + b + c + d + e + f &= 0 \\
2a + 2b + 2c + 2d - e - f &= 0 \\
3a + 3b - c - d - e - f &= 0
\end{aligned}
$$

The last column of the augmented matrix is zero all through the row reduction, so we need not write it in; we can work with the unaugmented matrix.

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & -1 & -1 \\ 3 & 3 & -1 & -1 & -1 & -1 \end{bmatrix}
\to
\begin{bmatrix} \color{blue}{1} & \color{red}{1} & \color{blue}{0} & \color{red}{0} & \color{blue}{0} & \color{red}{0} \\ \color{blue}{0} & \color{red}{0} & \color{blue}{1} & \color{red}{1} & \color{blue}{0} & \color{red}{0} \\ \color{blue}{0} & \color{red}{0} & \color{blue}{0} & \color{red}{0} & \color{blue}{1} & \color{red}{1} \end{bmatrix}
$$

$$
\color{blue}{a} + \color{red}{b} = 0 \qquad \color{blue}{c} + \color{red}{d} = 0 \qquad \color{blue}{e} + \color{red}{f} = 0.
$$

Move the independent variables (from non-pivot columns) to the RHS:

$$
\color{blue}{a} = -\color{red}{b} \qquad \color{blue}{c} = -\color{red}{d} \qquad \color{blue}{e} = -\color{red}{f}.
$$

If we prefer we can introduce new variables λ, μ and ν, and say that the general solution is

$$
\begin{array}{lll}
a = -\lambda & c = -\mu & e = -\nu \\
b = \lambda & d = \mu & f = \nu
\end{array}
$$

for arbitrary values of λ, μ and ν.