## Lecture 3

## Linear combinations

**Definition 7.1:** Let $v_1, \ldots, v_k$ and $w$ be vectors in $\mathbb{R}^n$. We say that $w$ is a *linear combination* of $v_1, \ldots, v_k$ if there exist scalars $\lambda_1, \ldots, \lambda_k$ such that

$$w = \lambda_1 v_1 + \cdots + \lambda_k v_k.$$

**Example 7.2:** Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \qquad w = \begin{bmatrix} 1 \\ 10 \\ 100 \\ -111 \end{bmatrix}$$

If we take $\lambda_1 = 1$ and $\lambda_2 = 11$ and $\lambda_3 = 111$ we get

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 111 \\ -111 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 100 \\ -111 \end{bmatrix} = w,$$

which shows that $w$ is a linear combination of $v_1$, $v_2$ and $v_3$.

## Linear combinations example

$w$ is a *linear combination* of $v_1, \ldots, v_k$ if there exist scalars $\lambda_1, \ldots, \lambda_k$ such that

$$w = \lambda_1 v_1 + \cdots + \lambda_k v_k.$$

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Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix} \qquad w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Any linear combination of $v_1, \ldots, v_4$ has the form

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = \begin{bmatrix} 0 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ 2\lambda_1 + 4\lambda_2 + 8\lambda_3 + 16\lambda_4 \\ \\ \end{bmatrix}.$$

Thus, the first component of any such linear combination is zero. (You should be able to see this without writing out the whole formula.) As the first component of $w$ is not zero, we see that $w$ is *not* a linear combination of $v_1, \ldots, v_4$.

## Linear combinations example

$w$ is a *linear combination* of $v_1, \ldots, v_k$ if there exist scalars $\lambda_1, \ldots, \lambda_k$ such that

$$w = \lambda_1 v_1 + \cdots + \lambda_k v_k.$$

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Consider the following vectors in $\mathbb{R}^3$:

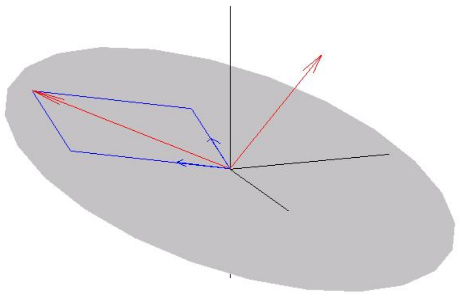
$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \quad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Any linear combination of $v_1, \ldots, v_5$ has the form

$$\lambda_1 v_1 + \cdots + \lambda_5 v_5 = \begin{bmatrix} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \end{bmatrix}.$$

The first and last components of any such linear combination are the same. Again, you should be able to see this without writing the full formula. As the first and last components of $w$ are different, we see that $w$ is not a linear combination of $v_1, \ldots, v_5$.

## Two vectors in $\mathbb{R}^3$ span a plane



Any vector that lies in the grey plane can be expressed as a linear combination of the two blue vectors.
Any vector that does not lie in the grey plane cannot be expressed as a linear combination of the two blue vectors.

## Method for finding linear combinations

Suppose we have vectors $v_1, \dots, v_k \in \mathbb{R}^n$ and another vector $w \in \mathbb{R}^n$, and we want to express $w$ as a linear combination of the $v_i$ (or show that this is not possible).
Let $A$ be the matrix whose columns are the vectors $v_i$:

$$A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_k \end{array}\right] \in M_{n\times k}(\mathbb{R}).$$

For any $k$-vector $\lambda = \begin{bmatrix}\lambda_1 & \cdots & \lambda_k\end{bmatrix}^T$ we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_k \end{array}\right] \begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_k\end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_k v_k$$

Thus, to express $w$ as a linear combination of the $v_i$ is the same as to solve the vector equation $A\lambda = w$, which we can do by row-reducing the augmented matrix

$$B = \left[\begin{array}{c|c} A & w \end{array}\right] = \left[\begin{array}{c|c|c|c} v_1 & \cdots & v_k & w \end{array}\right]$$

## Example of finding a linear combination

Is $w$ a linear combination of $v_1$, $v_2$ and $v_3$?

$$v_1 = \begin{bmatrix}11\\11\\1\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\11\\11\\1\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\11\\11\end{bmatrix} \qquad w = \begin{bmatrix}121\\221\\1211\\1111\end{bmatrix}.$$

We write down the relevant augmented matrix and row-reduce it:

$$\left[\begin{array}{ccc|c} 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \\ 1 & 1 & 11 & 1111 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 0 & -10 & -120 & -12100 \\ 0 & 0 & -120 & -12000 \\ & & & \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 11 & 1111 \\ 0 & 1 & 12 & 1210 \\ 0 & 0 & 1 & 100 \\ & & & \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 11 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 100 \\ & & & \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 100 \\ & & & \end{array}\right]$$

Move the bottom row to the top; Subtract multiples of row 1 from the other rows; Divide rows 2,3 and 4 by $-10$, $-120$ and 10; Subtract multiples of row 3 from the other rows; Subtract multiples of row 2 from the other rows.

## Example of finding a linear combination

$$v_1 = \begin{bmatrix}11\\11\\1\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}1\\11\\11\\1\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\11\\11\end{bmatrix} \qquad w = \begin{bmatrix}121\\221\\1211\\1111\end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 11 & 1 & 1 & 121 \\ 11 & 11 & 1 & 221 \\ 1 & 11 & 11 & 1211 \\ 1 & 1 & 11 & 1111 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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The final matrix corresponds to the system of equations

$$\lambda_1 = \quad \qquad \lambda_2 = \quad \qquad \lambda_3 = \quad \qquad 0 = 0$$

so we conclude that $w = \quad$.
In particular, $w$ can be expressed as a linear combination of $v_1$, $v_2$ and $v_3$.
We can check the above equation directly:

$$v_1 + 10v_2 + 100v_3 = \begin{bmatrix}11\\11\\1\\1\end{bmatrix} + \begin{bmatrix}\;\\\;\\\;\\\;\end{bmatrix} + \begin{bmatrix}100\\100\\1100\\1100\end{bmatrix} = \begin{bmatrix}121\\221\\1211\\1111\end{bmatrix} = w.$$

## Example of not finding a linear combination

Is $b$ a linear combination of $a_1$, $a_2$ and $a_3$?

$$a_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We write down the relevant augmented matrix and row-reduce it:

$$\left[\begin{array}{ccc|c} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 2 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 3 & 6 & 5 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 14 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Move the top row to the bottom, and multiply the other two rows by $-1$; Subtract 2 times row 1 from row 3; Subtract 3 times row 2 from row 3; Divide row 3 by 14; Subtract multiples of row 3 from rows 1 and 2.

## Example of not finding a linear combination

$$a_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ -1 & 0 & 3 & 2 \\ 0 & -1 & -2 & 3 \end{array}\right] \to \dots \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

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The final matrix has a pivot in the rightmost column, corresponding to the equation . This means that the equation $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = b$ cannot be solved for $\lambda_1$, $\lambda_2$ and $\lambda_3$, or in other words that $b$ is not a linear combination of $a_1$, $a_2$ and $a_3$.
We can also see this in a more direct but less systematic way, as follows. It is easy to check that $b.a_1 = b.a_2 = b.a_3 = 0$, which means that $b.(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) = 0$ for all possible choices of $\lambda_1$, $\lambda_2$ and $\lambda_3$. However, $b.b = 14 > 0$, so $b$ cannot be equal to $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3$.

## Linear independence

Definition 8.1: Let $\mathcal{V} = v_1, \dots, v_k$ be a list of vectors in $\mathbb{R}^n$.
A *linear relation* between the $v_i$ is a relation of the form $\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$, where $\lambda_1, \dots, \lambda_k$ are scalars.

For any list we have the trivial linear relation $0v_1 + 0v_2 + \cdots + 0v_k = 0$.
There may or may not be any nontrivial linear relations.

If $\mathcal{V}$ has a nontrivial linear relation, we say that it is *(linearly) dependent*.
If the only linear relation on $\mathcal{V}$ is the trivial one, we instead say that $\mathcal{V}$ is *(linearly) independent*.

Example 8.2: Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

There is a nontrivial linear relation $v_1 + v_2 - v_3 - v_4 = 0$, so the list $\mathcal{V}$ is .

## Linear dependence example

The list $v_1, \dots, v_k$ is *dependent* if there is a relation $\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$ with not all $\lambda_i$ being zero. Otherwise, it is *independent*.

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Example 8.3: Consider the list $\mathcal{A}$ given by

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Just by writing it out, you can check that $3a_1 + a_2 + 3a_3 - 4a_4 = 0$.
This is a nontrivial linear relation on the list $\mathcal{A}$, so $\mathcal{A}$ is .

Example 8.4: Claim: the following list $\mathcal{U}$ is independent.

$$u_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$$

Indeed, consider a linear relation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$. This gives

$$\begin{bmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 \\ \lambda_2 + \lambda_3 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \qquad \lambda_1 = 0; \qquad \lambda_3 = 0; \qquad \lambda_1 + \lambda_2 = 0; \qquad \lambda_2 = 0.$$

As the only linear relation is the trivial one, we see that $\mathcal{U}$ is .

## Pivots in every column

**Definition 8.6:** Let $B$ be a $p \times q$ matrix.
We say that $B$ is *wide* if $p < q$, or *square* if $p = q$ or *tall* if $p > q$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

**Lemma 8.7:** Let $B$ be a $p \times q$ matrix in RREF.

(a) If $B$ is wide then it is impossible for every column to contain a pivot.

(b) If $B$ is square and every column contains a pivot then $B =$ .

(c) If $B$ is tall then the only way for every column to contain a pivot is if $B$ consists of $I_q$ with $(p - q)$ rows of zeros added at the bottom.

$$B = \left[\frac{I_q}{0_{(p-q)\times q}}\right]$$

## Checking dependence by row-reduction

**Method 8.8:** Let $\mathcal{V} = v_1, \dots, v_m$ be a list of vectors in $\mathbb{R}^n$.
We can check whether this list is dependent as follows.

(a) Form the $n \times m$ matrix $A = \left[\begin{array}{c|c|c} v_1 & \dots & v_m \end{array}\right]$ whose columns are the vectors $v_i$.

(b) Row reduce $A$ to get another $n \times m$ matrix $B$ in RREF.

(c) If every column of $B$ contains a pivot (as on the previous slide) then $\mathcal{V}$ is independent.

(d) If some column of $B$ has no pivot, then the list $\mathcal{V}$ is dependent. Moreover, we can find the coefficients $\lambda_i$ in a nontrivial linear relation by solving the vector equation $B\lambda = 0$ (which is easy because $B$ is in RREF).

**Remark 8.9:** If $m > n$ then $\mathcal{V}$ is automatically dependent and need not do any more.
(Any list of 5 vectors in $\mathbb{R}^3$ is dependent, any list of 10 in $\mathbb{R}^9$ is dependent, $\dots$.)
Indeed, in this case $B$ is wide, so it cannot have a pivot in every column. This only tells us that there **exists** a nontrivial relation $\lambda_1 v_1 + \dots + \lambda_m v_m = 0$, it does not tell us the coefficients $\lambda_i$. To find them we do need to go through the whole method as explained above.

## Example of checking for (in)dependence

We previously considered the list

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\ & & & \\ & & & \\ & & & \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The end result has no pivot in the last column, so the original list is dependent. To find a specific linear relation, we solve the equation

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= -\lambda_4 \\ \lambda_2 &= -\lambda_4 \\ \lambda_3 &= \lambda_4 \\ \lambda_4 &\text{ arbitrary} \end{aligned}$$

Taking $\lambda_4 = 1$ gives $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1,$ $)$,
corresponding to the relation $-v_1$ $= 0$.

## Example of checking for (in)dependence

We previously considered the list

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \qquad a_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad a_4 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Here we have 4 vectors in $\mathbb{R}^2$, so they must be dependent. Thus, there exist nontrivial linear relations $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
To actually find such a relation, we write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix} 1 & \phantom{0} & \phantom{0} & \phantom{0} \\ 2 & & & \end{bmatrix} \to \begin{bmatrix} 1 & 12 & -1 & 3 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix}$$

We now need to solve the matrix equation

$$\begin{bmatrix} 1 & 0 & -11/23 & 9/23 \\ 0 & 1 & -1/23 & 5/23 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives $\lambda_1 = \frac{11}{23}\lambda_3 - \frac{9}{23}\lambda_4$ and $\lambda_2 = \frac{1}{23}\lambda_3 - \frac{5}{23}\lambda_4$ with $\lambda_3$ and $\lambda_4$ arbitrary.
If we choose $\lambda_3 = 23$ and $\lambda_4 = 0$ we get $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (11, 1, 23, 0)$ so we have a relation $= 0$.

## Example of checking for (in)dependence

We previously considered the list $\mathcal{U}$ given by

$$u_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad u_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} \qquad u_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}.$$

We can write down the corresponding matrix and row-reduce it as follows:

$$\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\\0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\ \hline 0&0&0\end{bmatrix}$$

The final matrix has a pivot in every column. It follows that the list $\mathcal{U}$ is ______.

## Proof of correctness of the method

Put $A = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right]$ as in step (a), and let $B$ be the RREF form of $A$.

Note that for any vector $\lambda = \begin{bmatrix}\lambda_1 & \dots & \lambda_m\end{bmatrix}^T \in \mathbb{R}^m$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] \begin{bmatrix}\lambda_1\\ \vdots \\ \lambda_m\end{bmatrix} = \lambda_1 v_1 + \cdots + \lambda_m v_m.$$

Thus, linear relations on our list are just the same as solutions to the homogeneous equation $A\lambda = 0$. We saw earlier that these are the same as solutions to the equation $B\lambda = 0$, which can be found by the standard method for RREF equations. If there is a pivot in every column then none of the variables $\lambda_i$ is independent, so the only solution is $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Thus, the only linear relation on $\mathcal{V}$ is the trivial one, which means that the list $\mathcal{V}$ is linearly independent.

Suppose instead that some column (the $k$'th one, say) does not contain a pivot. Then the variable $\lambda_k$ will be independent, so we can choose $\lambda_k = 1$. This will give us a nonzero to solution to $B\lambda = 0$, or equivalently $A\lambda = 0$, corresponding to a nontrivial linear relation on $\mathcal{V}$. This shows that $\mathcal{V}$ is linearly dependent.