Lecture 4

Spanning

Definition 9.1: Suppose we have a list $\mathcal{V} = v_1, \dots, v_m$ of vectors in $\mathbb{R}^n$. Then $\mathcal{V}$ *spans* $\mathbb{R}^n$ if **every** vector in $\mathbb{R}^n$ can be expressed as a linear combination of $v_1, \dots, v_m$.

Example 9.2: Consider the list $\mathcal{V} = v_1, v_2, v_3, v_4$, where

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

Previously we saw that the vector $w = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ is not a linear combination of this list, so the list $\mathcal{V}$ does not span $\mathbb{R}^4$.

Example 9.3: Consider the list $\mathcal{V} = v_1, v_2, v_3, v_4, v_5$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

Previously we saw that the vector $w = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$ is not a linear combination of this list, so the list $\mathcal{V}$ does not span $\mathbb{R}^3$.

Spanning example

Consider the list $\mathcal{U} = u_1, u_2, u_3$, where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

We will show that these span $\mathbb{R}^3$. Indeed, for any vector $v = \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3$ we can put

$$\lambda_1 = \frac{x+y-z}{2} \qquad \lambda_2 = \frac{x-y+z}{2} \qquad \lambda_3 = \frac{-x+y+z}{2}$$

and we find that

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = \begin{bmatrix} (x+y-z)/2 \\ (x+y-z)/2 \\ 0 \end{bmatrix} + \begin{bmatrix} (x-y+z)/2 \\ 0 \\ (x-y+z)/2 \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$= \begin{bmatrix} (x+y-z+x-y+z)/2 \\ (x+y-z-x+y+z)/2 \\ (x-y+z-x+y+z)/2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = v.$$

This expresses v as a linear combination of the list $\mathcal{U}$, as required.

Spanning example

Consider the list $\mathcal{A} = a_1, a_2, a_3$ where

$$a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad a_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^2$. Note that

$$(2y-4x)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (x-y)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2y-4x \\ 4y-8x \end{bmatrix} + \begin{bmatrix} 2x-2y \\ 3x-3y \end{bmatrix} + \begin{bmatrix} 3x \\ 5x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

or in other words

$$v = (\qquad)a_1 + (\qquad)a_2 + xa_3.$$

This expresses an arbitrary $v \in \mathbb{R}^2$ as a linear combination of a_1, a_2 and a_3, proving that the list $\mathcal{A}$ spans $\mathbb{R}^2$.

In this case there are actually many different ways in which we can express v as a linear combination of a_1, a_2 and a_3. Another one is

$$v = (y-3x)a_1 + (2x-2y)a_2 + ya_3.$$

Checking spanning by row-reduction

Method 9.7: Let $\mathcal{V} = v_1, \dots, v_m$ be a list of vectors in $\mathbb{R}^n$. We can check whether this list spans $\mathbb{R}^n$ as follows.

(a) Form the $m \times n$ matrix $C = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_m^T \end{array}\right]$ whose rows are the v_i^T.

(b) Row reduce C to get another $m \times n$ matrix D in RREF.

(c) If every column of D contains a pivot (so $D = \left[\begin{array}{c} I_n \\ \hline 0_{(m-n)\times n} \end{array}\right]$) then $\mathcal{V}$ spans $\mathbb{R}^n$.

(d) If some column of D has no pivot, then the list $\mathcal{V}$ does not span $\mathbb{R}^n$.

Remark 9.8: This is almost exactly the same as the method for checking independence, except that here we start by building a matrix C whose rows are v_i^T, instead of building a matrix A whose columns are v_i. These are transposes of each other: $A = C^T$ and $C = A^T$.

Warning: transposing does not interact well with row-reduction, so the matrix D is **not** the transpose of B.

Example of spanning check

Consider the list

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 16 \\ 81 \end{bmatrix}$$

The relevant matrix is $C = \begin{bmatrix} 0 & 1 & 2 & \\ 0 & 1 & 4 & \\ 0 & 1 & 8 & \\ & & & 81 \end{bmatrix}$

The first column is zero, and will remain zero no matter what row operations we perform. Thus C cannot reduce to the identity matrix, so $\mathcal{V}$ does not span (as we already saw by a different method). In fact the row-reduction is

$$C \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

but it is not really necessary to go through the whole calculation.

Example of spanning check

Consider the list $\mathcal{V}$ given by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \qquad v_5 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix} \to \left[\begin{array}{ccc} 1 & & 1 \\ 0 & & 0 \\ 0 & & 0 \\ \hline 0 & & 0 \\ 0 & & 0 \end{array}\right]$$

At the end of the process the last column does not contain a pivot (so the top 3×3 block is not the identity), so $\mathcal{V}$ does not span $\mathbb{R}^3$. Again, we saw this earlier by a different method.

Example of spanning check

For the list

$$\mathcal{A} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

the relevant row-reduction is

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 3 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 3 & -2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

In the last matrix the third column has no pivot, so the list does not span.

Example of spanning check

Consider the list $\mathcal{U} = u_1, u_2, u_3$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The relevant row-reduction is

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The end result is the identity matrix, so the list $\mathcal{U}$ spans $\mathbb{R}^3$.

Example of spanning check

Consider the list $\mathcal{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. The relevant row-reduction is

$$\begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \to \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right]$$

In the last matrix, the top 2×2 block is the identity. This means that the list $\mathcal{A}$ spans $\mathbb{R}^2$.

Proof of correctness of the method

Lemma 9.15: Let C be an $m \times n$ matrix, and let C' be obtained from C by a single elementary row operation. Let s be a row vector of length n. Then s can be expressed as a linear combination of the rows of C if and only if it can be expressed as a linear combination of the rows of C'.

Proof: Let the rows of C be $r_1, \dots, r_m$. Suppose that s is a linear combination of these rows, say

$$s = \lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 + \dots + \lambda_m r_m.$$

(a) Suppose that C' is obtained from C by swapping the first two rows, so the rows of C' are $r_2, r_1, r_3, \dots, r_m$. The sequence of numbers $\lambda_2, \lambda_1, \lambda_3, \dots, \lambda_m$ satisfies

$$s = \lambda_2 r_2 + \lambda_1 r_1 + \lambda_3 r_3 + \dots + \lambda_m r_m,$$

which expresses s as a linear combination of the rows of C'. The argument is essentially the same if we exchange any other pair of rows.

Proof of correctness of the method

$C \in M_{m \times n}(\mathbb{R})$; C' obtained from C by a single row operation; s a row vector of length n. Claim: s is a linear combination of rows of C iff it is a linear combination of rows of C'.

(b) Suppose instead that C' is obtained from C by multiplying the first row by a nonzero scalar u, so the rows of C' are $ur_1, r_2, \dots, r_m$. The sequence of numbers $u^{-1}\lambda_1, \lambda_2, \dots, \lambda_m$ then satisfies

$$s = (u^{-1}\lambda_1)(ur_1) + \lambda_2 u_2 + \dots + \lambda_m r_m,$$

which expresses s as a linear combination of the rows of C'.The argument is essentially the same if we multiply any other row by a constant.

(c) Suppose instead that C' is obtained from C by adding u times the second row to the first row, so the rows of C' are $r_1 + ur_2, r_2, r_3, \dots, r_m$. The sequence of numbers $\lambda_1, \lambda_2 - u\lambda_1, \lambda_3, \dots, \lambda_n$ then satisfies

$$\lambda_1(r_1 + ur_2) + (\lambda_2 - u\lambda_1)r_2 + \lambda_3 r_3 + \dots + \lambda_m r_m = \lambda_1 r_1 + \lambda_2 r_2 + \dots + \lambda_m r_m = s,$$

which expresses s as a linear combination of the rows of C'. The argument is essentially the same if add a multiple of any row to any other row.

Proof of correctness of the method

$C \in M_{m\times n}(\mathbb{R})$; C' obtained from C by a single row operation; s a row vector of length n. Claim: s is a linear combination of rows of C iff it is a linear combination of rows of C'.

We have now proved half of the lemma: if s is a linear combination of the rows of C, then it is also a linear combination of the rows of C'. We also need to prove the converse: if s is a linear combination of the rows of C', then it is also a linear combination of the rows of C. We will only treat case (c), and leave the other two cases as an exercise. The rows of C' are then $r_1 + ur_2, r_2, r_3, \dots, r_m$. As s is a linear combination of these rows, we have $s = \mu_1(r_1 + ur_2) + \mu_2 r_2 + \cdots + \mu_m r_m$ for some numbers $\mu_1, \dots, \mu_m$. Now the sequence of numbers $\mu_1, (\mu_2 + u\mu_1), \mu_3, \dots, \mu_m$ satisfies

$$s = \mu_1 r_1 + (\mu_2 + u\mu_1) r_2 + \mu_3 r_3 + \cdots + \mu_m r_m,$$

which expresses s as a linear combination of the rows of C.

Proof of correctness of the method

$C \in M_{m\times n}(\mathbb{R})$; C' obtained from C by a single row operation; s a row vector of length n. Claim: s is a linear combination of rows of C iff it is a linear combination of rows of C'.

Corollary 9.16: Let C be an $m \times n$ matrix, and let D be obtained from C by a sequence of elementary row operation. Let s be a row vector of length n. Then s can be expressed as a linear combination of the rows of C if and only if it can be expressed as a linear combination of the rows of D.

Proof.
Just apply the lemma to each step in the row-reduction sequence. □

Proof of correctness of the method

Lemma 9.17: Let D be an $m \times n$ matrix in RREF.

(a) Suppose that every column of D contains a pivot, so $D = \left[\frac{I_n}{0_{(m-n)\times n}}\right]$. Then every row vector of length n can be expressed as a linear combination of the rows of D.

(b) Suppose instead that the k'th column of D does not contain a pivot. Then the k'th standard basis vector e_k **cannot** be expressed as a linear combination of the rows of D.

Proof of (a): In this case the first n rows are the standard basis vectors

$$r_1 = e_1^T = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix}$$
$$r_2 = e_2^T = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\end{bmatrix}\cdots$$
$$r_n = e_n^T = \begin{bmatrix}0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

and $r_i = 0$ for $i > n$. This means that any row vector $v = \begin{bmatrix}v_1 & v_2 & \cdots & v_n\end{bmatrix}$ can be expressed as

$$\begin{aligned} v = &\begin{bmatrix}v_1 & 0 & 0 & \cdots & 0\end{bmatrix} + \\ &\begin{bmatrix}0 & v_2 & 0 & \cdots & 0\end{bmatrix} + \cdots + \\ &\begin{bmatrix}0 & 0 & 0 & \cdots & v_n\end{bmatrix} \\ = &v_1 r_1 + v_2 r_2 + v_3 r_3 + \cdots + v_n r_n, \end{aligned}$$

which is a linear combination of the rows of D.

Proof of correctness of the method

Lemma: Let D be an $m \times n$ matrix in RREF.

(a) Suppose that every column of D contains a pivot, so $D = \left[\frac{I_n}{0_{(m-n)\times n}}\right]$. Then every row vector of length n can be expressed as a linear combination of the rows of D.

(b) Suppose instead that the k'th column of D does not contain a pivot. Then the k'th standard basis vector e_k **cannot** be expressed as a linear combination of the rows of D.

Example for proof of (b): Consider the matrix

$$D = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is in RREF, with pivots in columns 2, 5 and 8. Let r_i be the i'th row, and consider a linear combination
$\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 = \begin{bmatrix}0 & \lambda_1 & 2\lambda_1 & 3\lambda_1 & \lambda_2 & 4\lambda_1 + 6\lambda_2 & 5\lambda_1 + 7\lambda_2 & \lambda_3\end{bmatrix}$.
The entries in the pivot columns 2, 5 and 8 of s are just the coefficients λ_1, λ_2 and λ_3. This is not a special feature of this example: it simply reflects the fact that pivot columns contain nothing except the pivot. Now consider

$$e_6 = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}$$

For this to be $\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3$ we need $\lambda_1 = 0$ and $\lambda_2 = 0$ and $\lambda_3 = 0$ (by looking in the pivot columns). But that means $\lambda_1 r_1 + \lambda_2 r_2 + \lambda_3 r_3 = 0 \neq e_6$.

Proof of correctness of the method

Lemma: Let D be an $m \times n$ matrix in RREF.

(b) Suppose instead that the k'th column of D does not contain a pivot. Then the k'th standard basis vector e_k **cannot** be expressed as a linear combination of the rows of D.

This line of argument works more generally.
Suppose that D is an RREF matrix and that the k'th column has no pivot.
We claim that e_k is not a linear combination of the rows of D.
We can remove any rows of zeros from D without affecting the question, so we may assume that every row is nonzero, so every row contains a pivot.
Suppose that $e_k = \lambda_1 r_1 + \cdots + \lambda_m r_m$ say.
By looking in the column that contains the first pivot, we see that $\lambda_1 = 0$.
By looking in the column that contains the second pivot, we see that $\lambda_2 = 0$.
Continuing in this way, we see that all the coefficients λ_i are zero, so $\sum_i \lambda_i r_i = 0$, which contradicts the assumption that $e_k = \lambda_1 r_1 + \cdots + \lambda_m r_m$.

We conclude that in fact it is impossible to write e_k as $\lambda_1 r_1 + \cdots + \lambda_m r_m$, so e_k is not a linear combination of the rows of D.

Duality

Consider an $n \times m$ matrix

$$P = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] = \left[\begin{array}{c} w_1^T \\ \hline \vdots \\ \hline w_n^T \end{array}\right] \in M_{n\times m}(\mathbb{R})$$

$$P^T = \left[\begin{array}{c|c|c} w_1 & \cdots & w_n \end{array}\right] = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_m^T \end{array}\right] \in M_{m\times n}(\mathbb{R})$$

- The vectors v_i are linearly independent in $\mathbb{R}^n$ if and only if $P \to \left[\begin{array}{c} I_m \\ \hline 0 \end{array}\right]$, if and only if the vectors w_j span $\mathbb{R}^m$.
- The vectors v_i span $\mathbb{R}^n$ if and only if $P^T \to \left[\begin{array}{c} I_n \\ \hline 0 \end{array}\right]$, if and only if the vectors w_j are linearly independent in $\mathbb{R}^m$.

In other words:

- The columns of P are independent if and only if the columns of P^T span.
- The columns of P span if and only if the columns of P^T are independent.