Lecture 5

Linear independence

Definition 8.1: Let $\mathcal{V} = v_1, \dots, v_k$ be a list of vectors in $\mathbb{R}^n$.
A *linear relation* between the v_i is a relation of the form
, where $\lambda_1, \dots, \lambda_k$ are scalars.

For any list we have the trivial linear relation .
There may or may not be any nontrivial linear relations.

If $\mathcal{V}$ has a nontrivial linear relation, we say that it is *(linearly)* .
If the only linear relation on $\mathcal{V}$ is the trivial one, we instead say that $\mathcal{V}$ is *(linearly)* .

Example 8.4: The following list $\mathcal{U}$ is :

$$u_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T \qquad u_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T \qquad u_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$$

Indeed, consider a linear relation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$. This gives

$$\begin{bmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 \\ \lambda_2 + \lambda_3 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad ; \quad ; \quad ; \quad .$$

As the only linear relation is the trivial one, we see that $\mathcal{U}$ is .

Checking dependence by row-reduction

Method 8.8: Let $\mathcal{V} = v_1, \dots, v_m$ be a list of vectors in $\mathbb{R}^n$.
We can check whether this list is dependent as follows.

(a) Form the $n \times m$ matrix $A = \left[\begin{array}{c|c|c} v_1 & \dots & v_m \end{array} \right]$ whose columns are the vectors v_i.
(b) Row reduce A to get another $n \times m$ matrix B in RREF.
(c) If every column of B contains a pivot (so $B = \left[\frac{I_m}{0_{(n-m)\times m}} \right]$) then $\mathcal{V}$ is independent.
(d) If some column of B has no pivot, then the list $\mathcal{V}$ is dependent.

Remark 8.9: If $m > n$ then B is wide, so there cannot be a pivot in every column, so $\mathcal{V}$ is automatically dependent and we need not do any more. (Any list of 5 vectors in $\mathbb{R}^3$ is dependent, any list of 10 vectors in $\mathbb{R}^9$ is dependent, and so on.)

Spanning

Definition 9.1: Suppose we have a list $\mathcal{V} = v_1, \dots, v_m$ of vectors in $\mathbb{R}^n$. Then $\mathcal{V}$ *spans* $\mathbb{R}^n$ if **every** vector in $\mathbb{R}^n$
$v_1, \dots, v_m$.

Example : Consider the list $\mathcal{U} = u_1, u_2, u_3$, where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For any vector $v = \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3$ we can put

$$\lambda_1 = \frac{x+y-z}{2} \qquad \lambda_2 = \frac{x-y+z}{2} \qquad \lambda_3 = \frac{-x+y+z}{2}$$

and we find that $v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$. This shows that $\mathcal{U}$ spans $\mathbb{R}^3$.

Checking spanning by row-reduction

Method 9.7: Let $\mathcal{V} = v_1, \dots, v_m$ be a list of vectors in $\mathbb{R}^n$. We can check whether this list spans $\mathbb{R}^n$ as follows.

(a) Form the $m \times n$ matrix $C = \left[\begin{array}{c} v_1^T \\ \hline \vdots \\ \hline v_m^T \end{array}\right]$ whose rows are the v_i^T.

(b) Row reduce C to get another $m \times n$ matrix D in RREF.

(c) If every column of D contains a pivot (so $D = \left[\begin{array}{c} I_n \\ \hline 0_{(m-n)\times n} \end{array}\right]$) then $\mathcal{V}$ spans $\mathbb{R}^n$.

(d) If some column of D has no pivot, then the list $\mathcal{V}$ does not span $\mathbb{R}^n$.

Remark 9.9: If $m < n$ then D is wide, so there cannot be a pivot in every column, so $\mathcal{V}$ cannot span and we need not do any more.
(No list of 3 vectors can span $\mathbb{R}^6$, no list of 10 vectors can span $\mathbb{R}^{42}$, and so on.)

Bases

Definition 10.1: A *basis* for $\mathbb{R}^n$ is a linearly independent list of vectors in $\mathbb{R}^n$ that also spans $\mathbb{R}^n$.

Remark 10.2: Any basis for $\mathbb{R}^n$ must contain precisely n vectors. Indeed, we saw before that a linearly independent list can contain at most n vectors, that a spanning list must contain at least n vectors. As a basis has both these properties, it must contain precisely n vectors.

Basis example

Consider the list $\mathcal{U} = (u_1, u_2, u_3)$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For an arbitrary vector $v = \begin{bmatrix} x & y & z \end{bmatrix}^T$ we have

$$(a-b)u_1 + (b-c)u_2 + cu_3 = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \\ \\ 0 \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = v,$$

which expresses v as a linear combination of u_1, u_2 and u_3. This shows that $\mathcal{U}$ spans $\mathbb{R}^3$. Now suppose we have a linear relation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$. This means that

$$\begin{bmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

from which we read off that $\lambda_3 = 0$, then that $\lambda_2 = 0$, then that $\lambda_1 = 0$. This means that the only linear relation on $\mathcal{U}$ is the trivial one, so $\mathcal{U}$ is linearly independent. As it also spans, we conclude that $\mathcal{U}$ is a basis.

Basis criterion

Proposition 10.4: Given $\mathcal{V} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$, put

$$A = \left[\begin{array}{c|c|c} v_1 & \dots & v_n \end{array}\right] \in M_{n\times n}(\mathbb{R})$$

Then $\mathcal{V}$ is a basis iff $A\lambda = x$ has a **unique** solution for every $x \in \mathbb{R}^n$. Proof: Suppose that $\mathcal{V}$ is a basis. In particular, this means that any vector $x \in \mathbb{R}^n$ can be expressed as a linear combination $x = \lambda_1 v_1 + \dots + \lambda_n v_n$.
Thus, if we form the vector $\lambda = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix}^T$, we have

$$A\lambda = \left[\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array}\right] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \lambda_1 v_1 + \dots + \lambda_n v_n = x,$$

so λ is a solution to $A\lambda = x$. Suppose that μ is also a solution, so

$$\mu_1 v_1 + \dots + \mu_n v_n = x.$$

By subtracting this from the earlier equation, we get

$$(\lambda_1 - \mu_1)v_1 + \dots + (\lambda_n - \mu_n)v_n = 0.$$

This is a linear relation on the independent list $\mathcal{V}$, so it must be the trivial one, so the coefficients $\lambda_i - \mu_i$ are zero, so $\lambda = \mu$. In other words, λ is the **unique** solution to $A\lambda = x$, as required.

Basis criterion

Proposition 10.4: Given $\mathcal{V} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$, put

$$A = \left[\begin{array}{c|c|c} v_1 & \dots & v_n \end{array}\right] \in M_{n\times n}(\mathbb{R})$$

Then $\mathcal{V}$ is a basis iff $A\lambda = x$ has a **unique** solution for every $x \in \mathbb{R}^n$.

We now need to prove the converse. Suppose that for every $x \in \mathbb{R}^n$, the equation $A\lambda = x$ has a unique solution. Equivalently, for every $x \in \mathbb{R}^n$, there is a unique sequence of coefficients $\lambda_1, \dots, \lambda_n$ such that $\lambda_1 v_1 + \dots + \lambda_n v_n = x$. Firstly, we can temporarily ignore the uniqueness, and just note that every element $x \in \mathbb{R}^n$ can be expressed as a linear combination of $v_1, \dots, v_n$. This means that the list $\mathcal{V}$ spans $\mathbb{R}^n$. Next, consider the case $x = 0$. The equation $A\lambda = 0$ has $\lambda = 0$ as one solution. By assumption, the equation $A\lambda = 0$ has a unique solution, so $\lambda = 0$ is the only solution. Using the standard equation for $A\lambda$, we can restate this as follows: the only sequence $(\lambda_1, \dots, \lambda_n)$ for which $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ is the sequence $(0, \dots, 0)$. In other words, the only linear relation on $\mathcal{V}$ is the trivial one. This means that $\mathcal{V}$ is linearly independent, and it also spans $\mathbb{R}^n$, so it is a basis.

Method to check for a basis

Let $\mathcal{V} = (v_1, \dots, v_m)$ be a list of vectors in $\mathbb{R}^n$.

(a) If $m \neq n$ then $\mathcal{V}$ is not a basis.

(b) If $m = n$ then we form the matrix

$$A = \left[\begin{array}{c|c|c} v_1 & \dots & v_m \end{array}\right]$$

and row-reduce it to get a matrix B.

(c) If $B = I_n$ then $\mathcal{V}$ is a basis; otherwise, it is not.

Proof:

(a) Has been discussed already: any basis of $\mathbb{R}^n$ has n vectors.

(b) If $A \to I_n$ then the same steps give $[A|x] \to [I_n|x']$, then $\lambda = x'$ is the unique solution to $A\lambda = x$. Thus $\mathcal{V}$ is a basis.

(c) If $A \to B \neq I_n$ then B cannot have a pivot in every column. By our method for checking independence, the list $\mathcal{V}$ is dependent and so is not a basis.

Basis example

Consider the vectors

$$v_1 = \begin{bmatrix}1\\2\\3\\2\\1\end{bmatrix} \qquad v_2 = \begin{bmatrix}3\\2\\1\\2\\3\end{bmatrix} \qquad v_3 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \qquad v_4 = \begin{bmatrix}1\\3\\5\\3\\1\end{bmatrix} \qquad v_5 = \begin{bmatrix}5\\3\\1\\3\\5\end{bmatrix}$$

To decide whether they form a basis, we construct the corresponding matrix A and start row-reducing it:

$$\begin{bmatrix}1&3&1&1&5\\2&&&&3\\3&&&&1\\2&&&&3\\1&3&1&1&5\end{bmatrix} \to \begin{bmatrix}1&3&1&1&5\\0&-4&-1&1&-7\\0&-8&-2&2&-14\\0&-4&-1&1&-7\\0&0&0&0&0\end{bmatrix} \to \begin{bmatrix}1&3&1&1&5\\0&-4&-1&1&-7\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

Already after the first step we have a row of zeros, and it is clear that we will still have a row of zeros after we complete the row-reduction, so A does not reduce to the identity matrix, so the vectors v_i do not form a basis.

Basis example

Consider the vectors

$$p_1 = \begin{bmatrix}1\\1\\11\\1\end{bmatrix} \qquad p_2 = \begin{bmatrix}1\\11\\1\\11\end{bmatrix} \qquad p_3 = \begin{bmatrix}1\\1\\1\\11\end{bmatrix} \qquad p_4 = \begin{bmatrix}1\\11\\11\\11\end{bmatrix}$$

To decide whether they form a basis, we construct the corresponding matrix A and row reduce it:

$$\begin{bmatrix}1&1&1&1\\1&11&1&11\\11&1&1&11\\1&11&11&11\end{bmatrix} \to \begin{bmatrix}1&1&1&1\\0&10&0&10\\0&-10&-10&0\\0&10&10&10\end{bmatrix} \to \begin{bmatrix}1&1&1&1\\0&1&0&1\\0&1&1&0\\0&1&1&1\end{bmatrix} \to$$

$$\begin{bmatrix}1&1&1&1\\0&1&0&1\\0&1&1&0\\0&0&0&1\end{bmatrix} \to \begin{bmatrix}1&1&1&1\\0&1&0&1\\0&0&1&-1\\0&0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

After a few more steps, we obtain ________. It follows that the list p_1, p_2, p_3, p_4 ______.

Coefficients in terms of a basis

Suppose that the list $\mathcal{V} = v_1, \dots, v_n$ is a basis for $\mathbb{R}^n$, and that w is another vector in $\mathbb{R}^n$. By the very definition of a basis, it must be possible to express w (in a unique way) as a linear combination $w = \lambda_1 v_1 + \cdots + \lambda_n v_n$. If we want to find the coefficients λ_i, we can use the following:

Method 10.8: Let $\mathcal{V} = v_1, \dots, v_n$ be a basis for $\mathbb{R}^n$, and let w be another vector in $\mathbb{R}^n$.

(a) Let B be the matrix

$$B = \left[\begin{array}{c|c|c|c} v_1 & \cdots & v_n & w \end{array}\right] \in M_{n\times(n+1)}(\mathbb{R}).$$

(b) Let B' be the RREF form of B. Then B' will have the form $[I_n|\lambda]$ for some column vector

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

(c) Now $w = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

It is clear from our recent discussion that this is valid.

Example of coefficients in terms of a basis

We will express $q = \begin{bmatrix} 0.9 & 0.9 & 0 & 10.9 \end{bmatrix}^T$ in terms of the basis p_1, p_2, p_3, p_4 in the previous example. We form the relevant augmented matrix, and apply the same row-reduction steps as before, except that we now have an extra column.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 1 & 11 & 1 & 11 & 0.9 \\ 11 & 1 & 1 & 11 & 0 \\ 1 & 11 & 11 & 11 & 10.9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 10 & 0 & 10 & 0 \\ 0 & -10 & -10 & 0 & -9.9 \\ 0 & 10 & 10 & 10 & 10 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0.99 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0.99 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0.9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0.99 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0.9 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0.99 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0.9 \\ 0 & 1 & 0 & 0 & -0.01 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \quad \\ 0 & 1 & 0 & 0 & \quad \\ 0 & 0 & 1 & 0 & \quad \\ 0 & 0 & 0 & 1 & 0.01 \end{array}\right]$$

The final result is $[I_4|\lambda]$, where $\lambda = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}^T$. This means that q can be expressed in terms of the vectors p_i as follows:

$$q = \quad p_1 - \quad p_2 + p_3 + \quad p_4.$$

Example of coefficients in terms of a basis

One can check that the vectors u_1, u_2, u_3 and u_4 below form a basis for $\mathbb{R}^4$.

$$u_1 = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 1/4 \\ 1/5 \\ 1/6 \\ 1/7 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We would like to express v in terms of this basis. The matrix formed by the vectors u_i is called the *Hilbert matrix*; it is notoriously hard to row-reduce. We will therefore use Maple.

Example of coefficients in terms of a basis

```
with(LinearAlgebra):
RREF := ReducedRowEchelonForm;
u[1] := <1,1/2,1/3,1/4>;
u[2] := <1/2,1/3,1/4,1/5>;
u[3] := <1/3,1/4,1/5,1/6>;
u[4] := <1/4,1/5,1/6,1/7>;
v    := <1,1,1,1>;
B    := <u[1]|u[2]|u[3]|u[4]|v>;
RREF(B);
```

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/3 & 1/4 & 1 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 60 \\ 0 & 0 & 1 & 0 & -180 \\ 0 & 0 & 0 & 1 & 140 \end{array}\right].$$

We conclude that

$$v = -4u_1 + 60u_2 - 180u_3 + 140u_4.$$

Duality for bases

Proposition 10.11: Let A be an $n \times n$ matrix. Then the columns of A form a basis for $\mathbb{R}^n$ if and only if the columns of A^T form a basis for $\mathbb{R}^n$.

Proof.
Recall:

- The colums of A span iff the columns of A^T are ________.
- The columns of A are independent iff the columns of A^T ________.
- A list is a basis iff ________

The claim is clear from this. □

Numerical criteria

Proposition 10.12: Let $\mathcal{V}$ be a list of n vectors in $\mathbb{R}^n$ (so the number of vectors is the same as the number of entries in each vector).

(a) If the list is linearly independent then it also spans, and so is a basis.
(b) If the list spans then it is also linearly independent, and so is a basis.

Proof.
Let A be the matrix whose columns are the vectors in $\mathcal{V}$.

(a) Suppose that $\mathcal{V}$ is linearly independent. Let B be the matrix obtained by row-reducing A. By the standard method for checking (in)dependence, B must have a pivot in every column. As B is also square, we must have ________. It follows that $\mathcal{V}$ is a basis.
(b) Suppose instead that $\mathcal{V}$ (which is the list of columns of A) spans $\mathbb{R}^n$. By duality, we conclude that the columns of A^T are linearly independent. Now A^T has n columns, so we can apply part (a) to deduce that the columns of A^T form a basis. By duality again, the columns of A must form a basis as well.

□