Lecture 7

Determinants

Definition : For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is defined as

$$\det(A) = \_\_\_\_ .$$

For a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ the determinant is defined by

$$\det(A) = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$
$$= \_\_\_\_ .$$

We will now discuss determinants for square matrices of any size. There are more details in an appendix to the printed notes, which will not be examined.

Determinants

Definition 12.1: Let A be an $n \times n$ matrix,
and let a_{ij} denote the entry in the i'th row of the j'th column.
We define

$$\det(A) = \_\_\_\_ ,$$

where the sum runs over all permutations σ of the set $\{1, \ldots, n\}$. Here $\mathrm{sgn}(\sigma)$ is the signature of σ. This means that $\mathrm{sgn}(\sigma) = +1$ if σ can be written as the product of an ____ number of transpositions, and $\mathrm{sgn}(\sigma) = -1$ otherwise.

One can check that this agrees with the standard formulae on the previous slide, if $n = 2$ or $n = 3$.

Determinants of triangular matrices

Example 12.4: Let A be an $n \times n$ matrix.

(a) If all the entries below the diagonal are zero, then the determinant is just the product of the diagonal entries: $\det(A) = \_\_\_\_ = \_\_\_\_$.
For example, we have

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1 \times 5 \times 8 \times 10 = 400.$$

(b) Similarly, if all the entries above the diagonal are zero, then the determinant is just the product of the diagonal entries.

(c) In particular, if A is a diagonal matrix (so all entries off the diagonal are zero) then both (a) and (b) apply and we have $\det(A) = \prod_{i=1}^{n} a_{ii}$.

(d) In particular, we have $\det(I_n) = \_\_$.

Basic facts about determinants

Example 12.5: If any row or column of A is zero, then $\det(A) = 0$.

Proposition 12.6: The determinants of elementary matrices are $\det(D_p(\lambda)) =$ and $\det(E_{pq}(\mu)) =$ and $\det(F_{pq}) =$.

$$D_2(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{24}(\mu) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Proposition 12.7: For any square matrix A, we have $\det(A^T) =$.

Theorem 12.8: If A and B are $n \times n$ matrices, then $\det(AB) =$.

Determinants and row operations

Method 12.9: Let A be an $n \times n$ matrix. We can calculate $\det(A)$ by applying row operations to A until we reach a matrix B for which we know $\det(B)$, keeping track of some factors as we go along.

(a) Every time we multiply a row by a number λ, we record the factor .

(b) Every time we exchange two rows, we record the factor .

Let μ be the product of these factors: then $\det(A) = \det(B)/\mu$.

Most obvious approach: continue until we reach B in RREF.

- If $B = I_n$ then $\det(B) = 1$ and $\det(A) = 1/\mu$.
- If $B \neq I_n$ then B must have a row of zeros so $\det(B) = 0$ and $\det(A) = 0$.

It will often be more efficient to stop the row-reduction at an earlier stage.

Example determinant by row-reduction

$$A = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 1 & 3 & 5 & 5 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{8}} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to$$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- Add multiples of row 4 to the other rows: no factor.
- Multiply each of the first three rows by $\frac{1}{2}$: overall factor of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
- Subtract row 1 from row 2: no factor.
- Subtract row 3 from row 2: no factor.
- Exchange rows 2 and 4: factor of -1.
- Exchange rows 1 and 2: another factor of -1.

The final matrix B is upper-triangular, so the determinant is just the product of the diagonal entries, which is $\det(B) = 2$. The product of the factors is $\mu = 1/8$, so $\det(A) = \det(B)/\mu = 16$.

Example determinant by row-reduction

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

- Subtract row 1 from each of the other rows: no factor.
- Subtract multiples of row 2 from rows 3 and 4: no factor.

As B has two rows of zeros, we see that $\det(B) = 0$.
The method therefore tells us that $\det(A) = \det(B)/\mu = 0$ as well.

Warning

Warning:

Most slides for this lecture have many transitions overlaying each other, so they cannot be printed in a useful way. Those slides have been omitted from this file. You should look at the printed notes and/or the version of the slides designed for online display instead.

Inverse of a Jordan block

Consider the matrix $P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The minor matrices are:

$$M_{11} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{13} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{14} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{22} = \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix} \quad M_{23} = \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix} \quad M_{24} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{33} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{34} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{41} = \begin{bmatrix} & 0 & 0 \\ & & 0 \\ & & \end{bmatrix} \quad M_{42} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad M_{43} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{44} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Each of these matrices is either upper triangular or lower triangular, so the determinant is the product of the diagonal entries.

Inverse of a Jordan block

Consider the matrix

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding minor determinants are as follows:

$$\begin{array}{llll} m_{11} = 1 & m_{12} = 0 & m_{13} = 0 & m_{14} = 0 \\ m_{21} = 1 & m_{22} = 1 & m_{23} = 0 & m_{24} = 0 \\ m_{31} = 1 & m_{32} = 1 & m_{33} = 1 & m_{34} = 0 \\ m_{41} = 1 & m_{42} = 1 & m_{43} = 1 & m_{44} = 1 \end{array}$$

and thus

$$\mathrm{adj}(P) = \begin{bmatrix} +m_{11} & -m_{21} & +m_{31} & -m_{41} \\ & & & \\ +m_{13} & -m_{23} & +m_{33} & -m_{43} \\ -m_{14} & +m_{24} & -m_{34} & +m_{44} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ & & & \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As P is upper triangular it is easy to see that $\det(P) = 1$ and so P^{-1} is the same as $\mathrm{adj}(P)$.